Inequalities for Convex and Non-Convex Functions

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Abstract

The paper discusses functions that are similar to convex functions, which may be convex but not necessarily. We consider the application of such functions to convex combinations with the common center, and sets with the common barycenter. The same functions are used in studying the integral quasi-arithmetic means.

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1. Introduction

In summary form, we present the concept of convexity and affinity using binomial combinations. Let $\mathcal{X}$ be a real linear space. Let $a, b \in \mathcal{X}$ be points and let $\alpha, \beta \in \mathbb{R}$ be coefficients. Their binomial combination $\alpha a + \beta b$ is convex if $\alpha, \beta \geq 0$ and if $\alpha + \beta = 1$. If $c = \alpha a + \beta b$, then the point $c$ itself is called the combination center.

A subset of $\mathcal{X}$ is convex if it contains all binomial convex combinations of its points. The convex hull $\text{conv}\mathcal{S}$ of a set $\mathcal{S} \subseteq \mathcal{X}$ is the smallest convex set which contains $\mathcal{S}$, and it consists of all binomial convex combinations of points of $\mathcal{S}$.

Let $\mathcal{C} \subseteq \mathcal{X}$ be a convex set. A function $f : \mathcal{C} \to \mathbb{R}$ is convex if the inequality $f(\alpha a + \beta b) \leq \alpha f(a) + \beta f(b)$ holds for all binomial convex combinations $\alpha a + \beta b$ of pairs of points $a, b \in \mathcal{C}$. 
Using the adjective affine instead of convex, requiring the coefficients condition \( \alpha + \beta = 1 \), and requiring the equality \( f(\alpha a + \beta b) = \alpha f(a) + \beta f(b) \), we get a characterization of the affinity.

Implementing mathematical induction, we can prove that all of the above applies to \( k \)-membered combinations for any positive integer \( k \).

We present the discrete and integral form of the famous Jensen’s inequality using convex and measurable sets.

In 1905, applying mathematical induction to convex combinations, Jensen (see [2]) has obtained the following.

**Discrete form of Jensen’s inequality.** Let \( \mathcal{C} \) be a convex set of a real linear space.

A convex function \( f : \mathcal{C} \to \mathbb{R} \) satisfies the inequality

\[
 f \left( \sum_{i=1}^{k} \alpha_i x_i \right) \leq \sum_{i=1}^{k} \alpha_i f(x_i) \tag{1.1}
\]

for every convex combination \( \sum_{i=1}^{k} \alpha_i x_i \) of points \( x_i \in \mathcal{C} \).

In 1906, working on transition to integrals, Jensen (see [3]) has stated the another form.

**Integral form of Jensen’s inequality.** Let \( \mathcal{A} \) be a measurable set of a space with positive measure \( \mu \) so that \( \mu(\mathcal{A}) > 0 \). Let \( \mathcal{I} \subseteq \mathbb{R} \) be an interval.

A convex function \( f : \mathcal{I} \to \mathbb{R} \) satisfies the inequality

\[
 f \left( \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} g(x) \, d\mu \right) \leq \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} f(g(x)) \, d\mu \tag{1.2}
\]

for every integrable function \( g : \mathcal{A} \to \mathbb{R} \) satisfying \( g(\mathcal{A}) \subseteq \mathcal{I} \), and providing that \( f(g) \) is integrable.

Numerous papers have been written on Jensen’s inequality, different types and variants can be found in [5], [6], [7] and [8].

2. **Discrete Inequalities**

This section is mostly a part of [7], and in this paper, it is an introduction to work with integrals. The most complete result is Theorem 2.3 relying on the idea of a convex function graph and its secant line.

Through the paper we will use an interval \( \mathcal{I} \subseteq \mathbb{R} \), and a bounded closed subinterval \( [a,b] \subseteq \mathcal{I} \) with endpoints \( a < b \).

Every number \( x \in \mathbb{R} \) can be uniquely presented as the binomial affine combination

\[
 x = \frac{b-x}{b-a} a + \frac{x-a}{b-a} b, \tag{2.1}
\]

which is convex if, and only if, the number \( x \) belongs to the interval \( [a,b] \).

Let \( \mathcal{I} \subseteq \mathbb{R} \) be an interval containing \( [a,b] \), let \( f : \mathcal{I} \to \mathbb{R} \) be a function,
and let $f_{\{a,b\}}^{\text{line}} : \mathbb{R} \to \mathbb{R}$ be the function of the secant line passing through the points $A(a, f(a))$ and $B(b, f(b))$ of the graph of $f$. Applying the affinity of the function $f_{\{a,b\}}^{\text{line}}$ to the combination in (2.1), we obtain its equation

$$f_{\{a,b\}}^{\text{line}}(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b).$$

(2.2)

The consequence of the representations in equations (2.1) and (2.2) is the fact that every convex function $f : \mathcal{I} \to \mathbb{R}$ satisfies the inequality

$$f(x) \leq f_{\{a,b\}}^{\text{line}}(x) \quad \text{for} \quad x \in [a, b],$$

(2.3)

and the reverse inequality

$$f(x) \geq f_{\{a,b\}}^{\text{line}}(x) \quad \text{for} \quad x \in \mathcal{I} \setminus [a, b].$$

(2.4)

**Lemma 2.1.** Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, let $[a, b] \subseteq \mathcal{I}$ be a bounded closed subinterval, let $\sum_{i=1}^{n} \alpha_i a_i$ be a convex combination of points $a_i \in [a, b]$, and let $\sum_{j=1}^{m} \beta_j b_j$ be a convex combination of points $b_j \in \mathcal{I} \setminus (a, b)$.

If the above convex combinations have the common center

$$\sum_{i=1}^{n} \alpha_i a_i = \sum_{j=1}^{m} \beta_j b_j,$$

(2.5)

then the inequality

$$\sum_{i=1}^{n} \alpha_i f(a_i) \leq \sum_{j=1}^{m} \beta_j f(b_j).$$

(2.6)

holds for every function $f : \mathcal{I} \to \mathbb{R}$ satisfying equations (2.3)-(2.4).

**Proof.** To simplify the computation we use the secant equation $f_{\{a,b\}}^{\text{line}}(x) = \kappa x + \lambda$. Respecting equations (2.3)-(2.4), and applying the affinity of $f_{\{a,b\}}^{\text{line}}$, we get

$$\sum_{i=1}^{n} \alpha_i f(a_i) \leq \sum_{i=1}^{n} \alpha_i (\kappa a_i + \lambda) = \kappa \sum_{i=1}^{n} \alpha_i a_i + \lambda$$

$$= \kappa \sum_{j=1}^{m} \beta_j b_j + \lambda = \sum_{j=1}^{m} \beta_j (\kappa b_j + \lambda)$$

(2.7)

$$\leq \sum_{j=1}^{m} \beta_j f(b_j)$$

finishing the proof.

Let us look to the relationship between the functions used above and convex functions. Thus, we present the following equivalence.

**Corollary 2.2.** A function $f : \mathcal{I} \to \mathbb{R}$ is convex if and only if it satisfies the implication (2.5) $\Rightarrow$ (2.6) for every bounded closed subinterval $[a, b] \subseteq \mathcal{I}$. 

Proof. The proof of necessity is covered by Lemma 2.1. The sufficiency will be proved via Jensen’s inequality. We take a convex combination

\[ c = \sum_{j=1}^{m} \beta_j b_j \]  

of different points \( b_j \in \mathcal{I} \) assuming that \( m \geq 2 \). Then the pair of points \( a = b_{j_1} < b_{j_2} = b \) exists so that \( c \in [a, b] \) and all \( b_j \in \mathcal{I} \setminus (a, b) \). The above combination admits one or two such pairs.

Recognizing the center \( c \) as one-membered convex combination \( 1c \), and using equation (2.10) as the implication premise, we get Jensen’s inequality

\[ f \left( \sum_{j=1}^{m} \beta_j b_j \right) = \sum_{j=1}^{m} \beta_j f(b_j) \]  

which guarantees the convexity of \( f \).

□

Let us finish the section by extending Lemma 2.1.

**Theorem 2.3.** Let \( \mathcal{I} \subseteq \mathbb{R} \) be an interval, let \([a, b] \subseteq \mathcal{I}\) be a bounded closed subinterval, let \( \sum_{i=1}^{n} \alpha_i a_i \) be a convex combination of points \( a_i \in [a, b] \), and let \( \sum_{j=1}^{m} \beta_j b_j \) be a convex combination of points \( b_j \in \mathcal{I} \setminus (a, b) \).

If the above convex combinations have the common center \( \alpha a + \beta b \), then the double inequality

\[ \sum_{i=1}^{n} \alpha_i f(a_i) \leq \alpha f(a) + \beta f(b) \leq \sum_{j=1}^{m} \beta_j f(b_j). \]  

holds for every function \( f : \mathcal{I} \to \mathbb{R} \) satisfying equations (2.3)-(2.4).

Proof. The series of inequalities in equation (2.7) should be extended with the middle term of the equality

\[ \kappa \sum_{i=1}^{n} \alpha_i a_i + \lambda = \alpha f(a) + \beta f(b) = \kappa \sum_{j=1}^{m} \beta_j b_j + \lambda. \]  

Then it yields the double inequality in equation (2.10). □

3. Main Results

The integral analogy of the concept of convex combination is the concept of barycenter. Let \( \mu \) be a positive measure on \( \mathbb{R} \). Let \( \mathcal{A} \subseteq \mathbb{R} \) be a \( \mu \)-measurable set with \( \mu(\mathcal{A}) > 0 \). Given the positive integer \( k \), let \( \mathcal{A} = \bigcup_{i=1}^{k} \mathcal{A}_{ki} \) be the partition of pairwise disjoint \( \mu \)-measurable sets \( \mathcal{A}_{ki} \). Taking points \( x_{ki} \in \mathcal{A}_{ki} \) we determine the convex combination

\[ x_k = \sum_{i=1}^{k} \frac{\mu(\mathcal{A}_{ki})}{\mu(\mathcal{A})} x_{ki} \]  

(3.1)
whose center \( x_k \) belongs to \( \text{conv} \mathcal{A} \). The limit \( M(\mathcal{A}, \mu) \) of the sequence \( (x_k)_k \) is the \( \mu \)-barycenter of the set \( \mathcal{A} \). So, the \( \mu \)-barycenter point

\[
M(\mathcal{A}, \mu) = \lim_{k \to \infty} \left( \sum_{i=1}^{k} \frac{\mu(\mathcal{A}_{ki})}{\mu(\mathcal{A})} x_{ki} \right) = \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} x \, d\mu.
\]

(3.2)

The barycenter of the set \( \mathcal{A} \) is in its convex hull, \( M(\mathcal{A}, \mu) \in \text{conv} \mathcal{A} \).

If \( g : \mathcal{A} \to \mathbb{R} \) is a \( \mu \)-integrable function, we similarly define its \( \mu \)-integral arithmetic mean by the number

\[
M_1(g, \mu) = \frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} g(x) \, d\mu.
\]

(3.3)

The integral arithmetic mean of the function \( g \) is in the convex hull of its image, \( M_1(g, \mu) \in \text{conv}\{g(\mathcal{A})\} \).

**Lemma 3.1.** Let \( \mu \) be a positive measure on \( \mathbb{R} \). Let \( \mathcal{I} \subseteq \mathbb{R} \) be an interval, let \([a, b] \subseteq \mathcal{I}\) be a bounded closed subinterval, and let \( \mathcal{A} \subseteq [a, b] \) and \( \mathcal{B} \subseteq \mathcal{I} \setminus (a, b) \) be sets of positive measures.

If the sets \( \mathcal{A} \) and \( \mathcal{B} \) have the common barycenter

\[
\frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} x \, d\mu = \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} x \, d\mu,
\]

(3.4)

then the inequality

\[
\frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} f(x) \, d\mu \leq \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} f(x) \, d\mu
\]

holds for every integrable function \( f : \mathcal{I} \to \mathbb{R} \) satisfying equations (2.3)-(2.4).

**Proof.** The calculation implemented to finite sums in equation (2.7) can also be applied to integrals. \( \square \)

**Corollary 3.2.** Let \( \mu \) be a positive measure on \( \mathbb{R} \). Let \( \mathcal{I} \subseteq \mathbb{R} \) be an interval, and let \( \mathcal{A} \subset \mathcal{B} \subseteq \mathcal{I} \) be an inclusion of a bounded interval \( \mathcal{A} \) and a subset \( \mathcal{B} \) of measures \( 0 < \mu(\mathcal{A}) < \mu(\mathcal{B}) \).

If the sets \( \mathcal{A} \) and \( \mathcal{B} \) have the common barycenter, then the double inequality

\[
\frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} f(x) \, d\mu \leq \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} f(x) \, d\mu \leq \frac{1}{\mu(\mathcal{B} \setminus \mathcal{A})} \int_{\mathcal{B} \setminus \mathcal{A}} f(x) \, d\mu
\]

(3.6)

holds for every integrable function \( f : \mathcal{I} \to \mathbb{R} \) satisfying equations (2.3)-(2.4) on the closure \([a, b]\) of the interval \( \mathcal{A} \).

**Proof.** The sets \( \mathcal{A} \subseteq [a, b] \) and \( \mathcal{B} \setminus \mathcal{A} \subseteq \mathcal{I} \setminus (a, b) \) have positive measures, and so they fulfill the conditions of Theorem 3.3. Since

\[
\frac{1}{\mu(\mathcal{A})} \int_{\mathcal{A}} f(x) \, d\mu = \frac{1}{\mu(\mathcal{B} \setminus \mathcal{A})} \int_{\mathcal{B} \setminus \mathcal{A}} x \, d\mu
\]

(3.7)
by equation (3.4), then it follows that
\[
\frac{1}{\mu(A)} \int_A f(x) \, d\mu \leq \frac{1}{\mu(B \setminus A)} \int_{B \setminus A} f(x) \, d\mu
\] (3.8)
by Lemma 3.1. To prove the whole inequality in equation (3.6), it remains to show that the middle term is the convex combination of end terms.

Using convex combination coefficients \( \alpha = \frac{\mu(A)}{\mu(B)} \) and \( \beta = \frac{\mu(B \setminus A)}{\mu(B)} \), we obtain the equality
\[
\frac{1}{\mu(B)} \int_B f(x) \, d\mu = \frac{1}{\mu(B)} \left( \int_A f(x) \, d\mu + \int_{B \setminus A} f(x) \, d\mu \right)
= \frac{\alpha}{\mu(A)} \int_A f(x) \, d\mu + \frac{\beta}{\mu(B \setminus A)} \int_{B \setminus A} f(x) \, d\mu
\] (3.9)
concluding the proof.

The integral analogy of Theorem 2.3 is as follows.

**Theorem 3.3.** Let \( \mu \) be a positive measure on \( \mathbb{R} \). Let \( I \subseteq \mathbb{R} \) be an interval, let \([a, b] \subseteq I\) be a bounded closed subinterval, and let \( A \subseteq [a, b] \) and \( B \subseteq I \setminus (a, b) \) be sets of positive measures.

If the sets \( A \) and \( B \) have the common barycenter \( \alpha a + \beta b \), then the double inequality
\[
\frac{1}{\mu(A)} \int_A f(x) \, d\mu \leq \alpha f(a) + \beta f(b) \leq \frac{1}{\mu(B)} \int_B f(x) \, d\mu
\] (3.10)
holds for every integrable function \( f : I \to \mathbb{R} \) satisfying equations (2.3)-(2.4).

Lemma 3.1 can be generalized by involving additional functions \( g \) and \( h \) in the following way.

**Corollary 3.4.** Let \( \mu \) be a positive measure on \( \mathbb{R} \). Let \( I \subseteq \mathbb{R} \) be an interval, let \([a, b] \subseteq I\) be a bounded closed subinterval, let \( A, B \subseteq \mathbb{R} \) be sets of positive measures, and let \( g : A \to \mathbb{R} \) and \( h : B \to \mathbb{R} \) be integrable functions such that \( g(A) \subseteq [a, b] \) and \( h(B) \subseteq I \setminus (a, b) \).

If the functions \( g \) and \( h \) have the common integral arithmetic mean
\[
\frac{1}{\mu(A)} \int_A g(x) \, d\mu = \frac{1}{\mu(B)} \int_B h(x) \, d\mu
\] (3.11)
then the inequality
\[
\frac{1}{\mu(A)} \int_A f(g(x)) \, d\mu \leq \frac{1}{\mu(B)} \int_B f(h(x)) \, d\mu
\] (3.12)
holds for every function \( f : I \to \mathbb{R} \) satisfying equations (2.3)-(2.4), and providing that \( f(g) \) and \( f(h) \) are integrable.

The above corollary will be used in applications to integral quasi-arithmetic means. It can be more generally stated by using measurable sets \( A \) and \( B \) of a space with positive measure \( \mu \) so that \( \mu(A) > 0 \) and \( \mu(B) > 0 \).
4. Applications to Quasi-Arithmetic Means

Functions investigated in this paper can be included to integral quasi-arithmetic means by applying methods such as those for convex functions.

Let $\mu$ be a positive measures on $\mathbb{R}$. Let $\mathcal{A} \subseteq \mathbb{R}$ be a measurable set of positive measure, and let $g : \mathcal{A} \to \mathbb{R}$ be a function. Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, and let $\varphi : \mathcal{I} \to \mathbb{R}$ be a strictly monotone continuous function such that $g(\mathcal{A}) \subseteq \mathcal{I}$, and the composite function $\varphi(g)$ is $\mu$-integrable on the set $\mathcal{A}$. The quasi-arithmetic mean of the function $g$ respecting the function $\varphi$ and measures $\mu$ can be defined by

$$M_\varphi(g, \mu) = \varphi^{-1} \left( \frac{1}{\mu(\mathcal{A})} \int_\mathcal{A} \varphi(g(x)) \, d\mu \right).$$  \hfill (4.1)

If $\mathcal{A}$ is the interval, then the above number is in $\mathcal{A}$. In that case, the term in parentheses belongs to the interval $\varphi(\mathcal{A})$, and therefore the quasi-arithmetic mean $M_\varphi(g, \mu)$ belongs to the interval $\mathcal{A}$.

In order to apply the convexity, we use strictly monotone continuous functions $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ such that $\psi$ is convex with respect to $\varphi$ (usually says $\psi$ is $\varphi$-convex), that is, the function $f = \psi(\varphi^{-1})$ is convex on the interval $\varphi(\mathcal{I})$.

We have the following application of Corollary 3.4 to quasi-arithmetic means.

**Theorem 4.1.** Let $\mu$ be a positive measure on $\mathbb{R}$. Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, let $[a, b] \subseteq \mathcal{I}$ be a bounded closed subinterval, let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}$ be sets of positive measures, and let $g : \mathcal{A} \to \mathbb{R}$ and $h : \mathcal{B} \to \mathbb{R}$ be integrable functions such that $g(\mathcal{A}) \subseteq [a, b]$ and $h(\mathcal{B}) \subseteq \mathcal{I} \setminus (a, b)$. Let $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ be strictly monotone continuous functions providing that $\varphi(g), \varphi(h), \psi(g), \psi(h)$ are integrable, and let $f = \psi(\varphi^{-1})$.

If $f$ satisfies equations (2.3)-(2.4) and $\psi$ is increasing, and if the equality

$$M_\varphi(g, \mu) = M_\varphi(h, \mu)$$  \hfill (4.2)

is valid, then we have the inequality

$$M_\psi(g, \mu) \leq M_\psi(h, \mu).$$  \hfill (4.3)

**Proof.** Taking $\mathcal{J} = \varphi(\mathcal{I})$, $[c, d] = \varphi([a, b])$, $u = \varphi(g)$ and $v = \varphi(h)$, we have $u(\mathcal{A}) \subseteq [c, d]$ and $v(\mathcal{B}) \subseteq \mathcal{J} \setminus (c, d)$. We will apply Corollary 3.4 to the functions $u : \mathcal{A} \to \mathbb{R}$, $v : \mathcal{B} \to \mathbb{R}$ and $f : \mathcal{J} \to \mathbb{R}$.

Using the equality $\varphi(M_\varphi(g, \mu)) = \varphi(M_\varphi(h, \mu))$ which follows from equation (4.2), and including the functions $u$ and $v$, we get

$$\frac{1}{\mu(\mathcal{A})} \int_\mathcal{A} u(x) \, d\mu = \frac{1}{\mu(\mathcal{B})} \int_\mathcal{B} v(x) \, d\mu.$$  \hfill (4.4)

Then it follows that

$$\frac{1}{\mu(\mathcal{A})} \int_\mathcal{A} f(u(x)) \, d\mu \leq \frac{1}{\mu(\mathcal{B})} \int_\mathcal{B} f(v(x)) \, d\mu$$  \hfill (4.5)
by Corollary 3.4. Applying the increasing function $\psi^{-1}$ to the above inequality, substituting $f(u(x))$ with $\psi(g(x))$, and $f(v(x))$ with $\psi(h(x))$, we obtain the inequality in equation (4.3).

All the cases of the above theorem are the following.

**Corollary 4.2.** Let the conditions of Theorem 4.1 be fulfilled.

If either $f$ satisfies equations (2.3)-(2.4) and $\psi$ is increasing or $-f$ satisfies equations (2.3)-(2.4) and $\psi$ is decreasing, and if the equality in equation (4.2) is valid, then the inequality in equation (4.3) holds.

If either $f$ satisfies equations (2.3)-(2.4) and $\psi$ is decreasing or $-f$ satisfies equations (2.3)-(2.4) and $\psi$ is increasing, and if the equality in equation (4.2) is valid, then the reverse inequality in equation (4.3) holds.

A special case of the quasi-arithmetic means in equation (4.1) are power means depending on real exponents $r$. Thus, using the functions

$$
\varphi_r(x) = \begin{cases} 
x^r & r \neq 0 \\
\ln x & r = 0
\end{cases}
$$

where $x \in (0, \infty)$, we get the power means of order $r$ in the form

$$
M_r(g, \mu) = \begin{cases} 
\left( \frac{1}{\mu(A)} \int_A (g(x))^r \, d\mu \right)^{\frac{1}{r}} & r \neq 0 \\
\exp \left( \frac{1}{\mu(A)} \int_A \ln g(x) \, d\mu \right) & r = 0
\end{cases}
$$

To apply Theorem 4.1 to the power means we use the interval of positive real numbers $I = (0, \infty)$.

**Corollary 4.3.** Let $\mu$ be a positive measure on $\mathbb{R}$. Let $I = (0, \infty)$, let $[a, b] \subseteq I$ be a bounded closed interval, let $A, B \subseteq \mathbb{R}$ be sets of positive measures, and let $g : A \to \mathbb{R}$ and $h : B \to \mathbb{R}$ be integrable functions such that $g(A) \subseteq [a, b]$ and $h(B) \subseteq I \setminus (a, b)$.

If

$$
M_1(g, \mu) = M_1(h, \mu),
$$

then

$$
M_r(g, \mu) \leq M_r(h, \mu) \quad \text{for } r \geq 1
$$

and

$$
M_r(g, \mu) \geq M_r(h, \mu) \quad \text{for } r \leq 1.
$$

**Proof.** The proof follows from Theorem 4.1 and Corollary 4.2 by applying the well-known convex and concave functions such as $\varphi(x) = x$ and $\psi(x) = x^r$ for $r \neq 0$, and $\psi(x) = \ln x$ for $r = 0$.

The basic facts relating to quasi-arithmetic and power means can be found in [1]. For more details on different forms of quasi-arithmetic and power means, as well as their refinements, see [4].
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