On the Simpson and Hermite-Hadamard Type Inequalities for the Geometrically Convex Functions

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Abstract

In this article, we establish some generalized Simpson type and Hermite-Hadamard type integral inequalities for differentiable functions whose \( q \)-th powers of absolute values of derivatives are monotonically decreasing and geometrically convex.

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1 Introduction

Recall that a function \( f : I \subseteq R \to R \) is said to be convex on \( I \) if the inequality

\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)
\]

holds for all \( x, y \in I \) and \( t \in [0, 1] \), and \( f \) is said to be concave on \( I \) if the inequality (3) holds in reversed direction.

The following double inequality is well known in the literature as Hermite-Hadamard type inequality: Let \( f : I \subseteq R \to R \) be a convex function define
on an interval $I$ of real numbers, and $a, b \in I$ with $a < b$. Then the following double inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{f(a) + f(b)}{2}.$$ \hspace{1cm} (1)

Both inequalities hold in the reversed direction if $f$ is concave.

It was first discovered by Hermite in 1881 in the Journal Mathesis. This inequality (1) was nowhere mentioned in the mathematical literature until 1893. In [2], Beckenbach, a leading expert on the theory of convex functions, wrote that the inequality (1) was proved by Hadamard in 1893. In 1974, Mitrinović found Hermite and Hadamard’s note in Mathesis. That is why, the inequality (1) was known as Hermite-Hadamard inequality.

Recall that the following inequality is well-known in the literature as Simpson type inequality: Let $f : [a, b] \rightarrow R$ be a four times continuously differentiable function on $(a, b)$ and $\| f^{(4)} \|_{\infty} = \sup_{x \in (a, b)} | f^{(4)}(x) | < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left\{ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right\} \right| \leq \frac{1}{2880} \| f^{(4)} \|_{\infty} (b-a)^4.$$ \hspace{1cm} (2)

For recent results and generalizations of Hermite-Hadamard inequality and Simpson inequality concerning convex functions, you may see [2, 3, 4, 5, 7, 18, 19, 20] and the references therein.

In [24], Zhang et al. introduced the concept of geometrically convex functions as the following:

**Definition 1.** A function $f : [0, b] \rightarrow R_+$ is said to be geometrically convex if the following inequality

$$f(x^t y^{1-t}) \leq [f(x)]^t [f(y)]^{1-t}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

For some recent results connected with geometrically convex functions, you may see [1, 3, 9, 10, 11, 12, 14, 15, 16, 17, 18, 22, 23, 24].

In [12], Özdemir et al. established the following theorems for continuously twice differentiable functions whose $q$-th powers of absolute values of derivatives are monotonically decreasing and geometrically convex:
Theorem 1.1. Let \( f : I \to \mathbb{R}_+ \) be a twice differentiable function on the interior \( I^0 \) of an interval \( I \) such that \( f'' \in L_1([a,b]) \), where \( a, b \in I \) with \( a < b \). If \( |f''|^q \) is monotonically decreasing and geometrically convex on \([a,b]\), for \( q > 1 \), then the following inequality

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right|
\leq \frac{(b-a)^2}{2} \left( \frac{\frac{3}{2} \Gamma(1+p)}{21+2p \Gamma\left(\frac{3}{2}+p\right)} \right)^{\frac{1}{2p}} L^{\frac{1}{2p}}\left(|f''(a)|^q, |f''(b)|^q\right)
\]

holds for \( 1 < p < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), where \( L(,) \) is logarithmic mean defined by

\[
L(a,b) = \frac{b-a}{\ln |b| - \ln |a|}.
\]

Lemma 1. For \( a, b \in [0, \infty) \) and \( m, t \in (0, 1] \), if \( a < b \) and \( b \geq 1 \), then the inequality holds:

\[
a^tb^{1-t} \leq ta + (1-t)b
\]

In this article, we establish some generalized Simpson type and Hermite-Hadamard type integral inequalities for differentiable functions whose \( q \)-th powers of absolute values of derivatives are monotonically decreasing and geometrically convex.

# 2 Simpson type integral inequalities

In this section, for the simplicity of the notation, let

\[
S_a^b(f)(h,n) \equiv \frac{1}{n}\{f(a) + (n-2)f(hb + (1-h)a) + f(b)\} - \frac{1}{b-a} \int_a^b f(x) \, dx
\]

for any integer \( n \geq 2 \) and \( h \in (0, 1) \) with \( \frac{1}{n} \leq h \leq \frac{n-1}{n} \).

Firstly, in order to generalize the Simpson type integral inequality, we need the following lemma:

Lemma 2. [13] Let \( f : I \to \mathbb{R} \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) in \( \mathbb{R}_+ \) such that \( f' \in L_1[a,b] \), where \( a, b \in I \) with \( a < b \). Then, for any integer \( n \geq 2 \) and \( h \in (0, 1) \) with \( \frac{1}{n} \leq h \leq \frac{n-1}{n} \), the following identity

\[
S_a^b(f)(h,n) = (b-a) \int_0^1 p(t,h) f'(tb + (1-t)a) \, dt
\]
for any $t \in [0,1]$, where

$$p(t, h) = \begin{cases} t - \frac{1}{n}, & t \in [0, h], \\ t - \frac{n-1}{n}, & t \in (h, 1]. \end{cases}$$

We will use the above lemma for obtaining several following results for differentiable geometrically convex and monotonically decreasing functions:

**Theorem 2.1.** Let $f : I \to \mathbb{R}$ be a differentiable function on the interior $I^0$ of an interval $I$ in $\mathbb{R}_+$ such that $f' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is geometrically convex and monotonically decreasing on $[a, b]$, then, for $n \geq 2$ and $h \in (0, 1)$ with $\frac{1}{n} \leq h \leq \frac{n-1}{n}$, the following inequality holds:

$$|S_b^h(f)(h, n)| \leq (b - a) \left| f'(a) \right| \left\{ \mu_{21}(M, 0, \frac{1}{n}) + \mu_{21}(M, h, \frac{1}{n}) + \mu_{21}(M, h, \frac{n-1}{n}) + \mu_{21}(M, 1, \frac{n-1}{n}) \right\},$$

where

$$M = \frac{|f'(b)|}{|f'(a)|}, \quad \mu_{21}(M, \alpha, \beta) = \frac{M^\beta - M^\alpha}{(\ln M)^2} + \frac{M^\alpha (\alpha - \beta)}{\ln M}.$$

**Proof.** From Lemma 1, we have

$$|S_b^h(f)(h, n)| = (b - a) \int_0^1 |p(t, h)||f'(tb + (1-t)a)| \, dt$$

$$= (b - a) \left[ \int_0^{\frac{1}{n}} \left( \frac{1}{n} - t \right) |f'(tb + (1-t)a)| \, dt + \int_{\frac{1}{n}}^h \left( t - \frac{1}{n} \right) |f'(tb + (1-t)a)| \, dt + \int_{h}^{\frac{n-1}{n}} \left( \frac{n-1}{n} - t \right) |f'(tb + (1-t)a)| \, dt + \int_{\frac{n-1}{n}}^1 \left( t - \frac{n-1}{n} \right) |f'(tb + (1-t)a)| \, dt \right].$$

(3)

Since $|f'|$ is geometrically convex and monotonically decreasing on $[a, b]$,
we have

\[
(i) \int_0^{\frac{1}{n}} \left( \frac{1}{n} - t \right) \left| f'(tb + (1 - t)a) \right| \, dt \\
\leq \int_0^{\frac{1}{n}} \left( \frac{1}{n} - t \right) \left| f'(a^{1-t}b) \right| \, dt \\
\leq \left| f'(a) \right| \int_0^{\frac{1}{n}} \left( \frac{1}{n} - t \right) \left( \frac{1}{\left| f'(a) \right|} \right) \, dt \\
= \left| f'(a) \right| \mu_{21}(M, 0, \frac{1}{n}), 
\]

(4)

\[
(ii) \int_{\frac{1}{n}}^{h} \left( t - \frac{1}{n} \right) \left| f'(tb + (1 - t)a) \right| \, dt \\
\leq \left| f'(a) \right| \int_{\frac{1}{n}}^{h} \left( t - \frac{1}{n} \right) \left( \frac{1}{\left| f'(a) \right|} \right) \, dt \\
= \left| f'(a) \right| \mu_{21}(M, h, \frac{1}{n}).
\]

(5)

\[
(iii) \int_{h}^{\frac{n-1}{n}} \left( \frac{n-1}{n} - t \right) \left| f'(tb + (1 - t)a) \right| \, dt \\
\leq \left| f'(a) \right| \int_{h}^{\frac{n-1}{n}} \left( \frac{n-1}{n} - t \right) \left( \frac{1}{\left| f'(a) \right|} \right) \, dt \\
= \left| f'(a) \right| \mu_{21}(M, h, \frac{n-1}{n}).
\]

(6)

\[
(iv) \int_{\frac{n-1}{n}}^{1} \left( t - \frac{n-1}{n} \right) \left| f'(tb + (1 - t)a) \right| \, dt \\
\leq \left| f'(a) \right| \int_{\frac{n-1}{n}}^{1} \left( \frac{n-1}{n} - t \right) \left( \frac{1}{\left| f'(a) \right|} \right) \, dt \\
= \left| f'(a) \right| \mu_{21}(M, 1, \frac{n-1}{n}).
\]

(7)

By substituting (4)-(7) in (3), we get the desired result.

**Theorem 2.2.** Let \( f : I \to R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) in \( R^+ \) such that \( f' \in L_1[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is geometrically convex and monotonically decreasing on \([a,b] \), for \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then, for \( n \geq 2 \) and \( h \in (0,1) \) with \( \frac{1}{n} \leq h \leq \frac{n-1}{n} \), the following
inequality holds:
\[
\left| S_a^b(f)(h, n) \right| \leq (b - a) |f'(a)| \left[ \left\{ \frac{1 + (nh - 1)^{1+p}}{n^{1+p}(1 + p)} \right\}^{\frac{q}{q}} \mu_{22}^{\frac{1}{q}}(M, 0, h) \right.
+ \left\{ \frac{1 + (n - nh - 1)^{1+p}}{n^{1+p}(1 + p)} \right\}^{\frac{q}{q}} \mu_{22}^{\frac{1}{q}}(M, h, 1) \right].
\]

where
\[
\mu_{22}(M, \alpha, \beta) = \frac{M^{q\beta} - M^{q\alpha}}{q \ln M}.
\]

Proof. From Lemma 1 and using the Hölder integral inequality, we have
\[
\left| S_a^b(f)(h, n) \right| = (b - a) \left[ \left( \int_0^h \left| t - \frac{1}{n} \right|^p dt \right)^{\frac{q}{q}} \left( \int_0^h \left| f'(tb + (1-t)a) \right|^q dt \right)^{\frac{1}{q}} \right.
+ \left( \int_h^1 \left| t - \frac{n-1}{n} \right|^p dt \right)^{\frac{q}{q}} \left( \int_h^1 \left| f'(tb + (1-t)a) \right|^q dt \right)^{\frac{1}{q}} \right]. \quad (8)
\]

Note that
\[
\int_0^h \left| t - \frac{1}{n} \right|^p dt = \frac{1 + (nh - 1)^{1+p}}{n^{1+p}(1 + p)}; \quad (9)
\int_h^1 \left| t - \frac{n-1}{n} \right|^p dt = \frac{1 + (n - nh - 1)^{1+p}}{n^{1+p}(1 + p)}. \quad (10)
\]

Since \( |f'| \) is geometrically convex and monotonically decreasing on \([a, b]\), by Lemma 1 we have
\[a^{1-t}b^t \leq tb + (1-t)a\] and
\[
\left| f'(tb + (1-t)a) \right|^q \leq \left| f'(a^{1-t}b^t) \right|^q. \quad (11)
\]

By (11), we have
\[
(i) \int_0^h \left| f'(tb + (1-t)a) \right|^q dt \leq \int_0^h \left| f'(a^{1-t}b^t) \right|^q dt \\
\leq \left| f'(a) \right|^q \int_0^h M^q dt \\
= \left| f'(a) \right|^q \mu_{22}(M, 0, h), \quad (12)
\]
\[
(ii) \int_h^1 \left| f'(tb + (1-t)a) \right|^q dt \leq \left| f'(a) \right|^q \mu_{22}(M, h, 1). \quad (13)
\]
By substituting (9), (10), (12) and (13) in (8), we get the desired result.

**Theorem 2.3.** Let \( f : I \rightarrow \mathbb{R} \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) in \( \mathbb{R}_+ \) such that \( f' \in L_1[a, b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is geometrically convex and monotonically decreasing on \([a, b]\), for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then, for \( n \geq 2 \) and \( h \in (0, 1) \) with \( \frac{1}{n} \leq h \leq \frac{n-1}{n} \), the following inequality holds:

\[
\left| S^b_a(f)(h,n) \right| \\
\leq (b-a) |f'(a)| \left\{ \left( 1 + \frac{(nh-1)^2}{2n^2} \right)^\frac{1}{p} \left\{ \mu_{23}(M, \frac{1}{n}, 0) + \mu_{23}(M, \frac{1}{n}, h) \right\}^\frac{1}{q} \\
+ \left( 1 + \frac{(n-nh-1)^2}{2n^2} \right)^\frac{1}{p} \left\{ \mu_{23}(M, \frac{n-1}{n}, h) + \mu_{23}(M, \frac{n-1}{n}, 1) \right\}^\frac{1}{q} \right\},
\]

where

\[
\mu_{23}(M, \alpha, \beta) = \frac{M^{\alpha q} - M^{\beta q}}{(q \ln M)^2} + \frac{(\beta - \alpha)M^{\beta q}}{q \ln M}.
\]

**Proof.** From Lemma 1 and using the power mean integral inequality, we have

\[
\left| S^b_a(f)(h,n) \right| \\
= (b-a) \left[ \left( \int_0^h |t - \frac{1}{n}| dt \right)^\frac{1}{p} \left( \int_0^h |t - \frac{1}{n}| |f'(tb + (1-t)a)|^q dt \right)^\frac{1}{q} \\
+ \left( \int_h^1 |t - \frac{n-1}{n}| dt \right)^\frac{1}{p} \left( \int_h^1 |t - \frac{n-1}{n}| |f'(tb + (1-t)a)|^q dt \right)^\frac{1}{q} \right].
\]

Note that

\[
\int_0^h |t - \frac{1}{n}| dt = \frac{1 + (nh-1)^2}{2n^2},
\]

\[
\int_h^1 |t - \frac{n-1}{n}| dt = \frac{1 + (n-nh-1)^2}{2n^2}.
\]

Since \( |f'|^q \) is geometrically convex and monotonically decreasing on \([a, b]\) for \( q \geq 1 \), we have

\[
(i) \int_0^h |t - \frac{1}{n}| |f'(tb + (1-t)a)|^q dt \\
= \int_0^{\frac{1}{n}} \left( \frac{1}{n} - t \right) |f'(tb + (1-t)a)|^q dt
\]
\[ + \int_{\frac{1}{n}}^{h} \left( t - \frac{1}{n} \right) |f'(tb + (1-t)a)|^q \, dt \]

\[ \leq \int_{0}^{\frac{1}{n}} \left( \frac{1}{n} - t \right) |f'(a^{1-t}b')|^q \, dt + \int_{\frac{1}{n}}^{h} \left( t - \frac{1}{n} \right) |f'(a^{1-t}b')|^q \, dt \]

\[ \leq \int_{0}^{\frac{1}{n}} \left( \frac{1}{n} - t \right) \left( |f'(a)|^{1-t} |f'(b)|^t \right)^q \, dt \]

\[ + \int_{\frac{1}{n}}^{h} \left( t - \frac{1}{n} \right) \left( |f'(a)|^{1-t} |f'(b)|^t \right)^q \, dt \]

\[ = |f'(a)|^q \left\{ \mu_{23}(M, \frac{1}{n}, 0) + \mu_{23}(M, \frac{1}{n}, h) \right\}, \quad (17) \]

\[ (ii) \int_{\frac{1}{n}}^{1} \left| t - \frac{n-1}{n} \right| |f'(tb + (1-t)a)|^q \, dt \]

\[ \leq \int_{\frac{1}{n}}^{1} \left( \frac{n-1}{n} - t \right) |f'(a^{1-t}b')|^q \, dt \]

\[ + \int_{\frac{1}{n}}^{1} \left( t - \frac{n-1}{n} \right) |f'(a^{1-t}b')|^q \, dt \]

\[ = |f'(a)|^q \left\{ \mu_{23}(M, \frac{n-1}{n}, h) + \mu_{23}(M, \frac{n-1}{n}, 1) \right\}, \quad (18) \]

By substituting (15)-(18) in (14), we get the desired result.

### 3 Hermite-Hadamard type inequality

In order to generalize the Hermite-Hadamard type integral inequalities for differentiable geometrically convex and monotonically decreasing functions, we need the following lemma:

**Lemma 3.** [6] Let \( f : I \to R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) in \( R_+ \) and \( a, b \in I \) with \( a < b \). If \( f' \in L[a,b] \) and \( \lambda, \nu \in [0, \infty) \) with \( \lambda + \nu > 0 \), then the following equality holds:

\[
\frac{\lambda f(a) + \nu f(b)}{\lambda + \nu} - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx = \frac{b-a}{\lambda + \nu} \int_{0}^{1} \left\{ (\lambda + \nu) t - \lambda \right\} f'(tb + (1-t)a) \, dt.
\]

**Theorem 3.1.** Let \( f : I \to R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) in \( R_+ \) such that \( f' \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is geometrically convex and monotonically decreasing on \([a,b]\), for \( q \geq 1 \), then,
for \( \lambda, \nu \in [0, \infty) \) with \( \lambda + \nu > 0 \) the following inequality holds:

\[
\left| \frac{\lambda f(a) + \nu f(b)}{\lambda + \nu} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \left| f'(a) \right| \left( \frac{b - a}{\lambda + \nu} \right) \left( \frac{\lambda^2 + \nu^2}{2(\lambda + \nu)} \right)^{1 - \frac{1}{q}} \\
\times \left[ \left\{ \lambda \mu_{31}(M, \frac{\lambda}{\lambda + \nu}, 0) - (\lambda + \nu) \mu_{32}(M, 0, \frac{\lambda}{\lambda + \nu}) \right\} \\
+ \left\{ (\lambda + \nu) \mu_{32}(M, \frac{\lambda}{\lambda + \nu}, 1) - \lambda \mu_{31}(M, 1, \frac{\lambda}{\lambda + \nu}) \right\} \right]^{\frac{1}{q}},
\]

where

\[
\mu_{31}(M, \alpha, \beta) = \frac{M^{\alpha q} - M^{\beta q}}{q \ln M},
\]

\[
\mu_{32}(M, \alpha, \beta) = \frac{M^{\alpha q} - M^{\beta q}}{(q \ln M)^2} + \frac{\beta M^{\beta q} - \alpha M^{\alpha q}}{q \ln M}.
\]

**Proof.** From Lemma 2 and using the power mean integral inequality, we have

\[
\left| \frac{\lambda f(a) + \nu f(b)}{\lambda + \nu} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b - a}{\lambda + \nu} \int_0^1 \left| (\lambda + \nu)t - \lambda \right| f'(tb + (1 - t)a) \, dt \\
\leq \frac{b - a}{\lambda + \nu} \left( \int_0^1 \left| (\lambda + \nu)t - \lambda \right| \, dt \right)^{1 - \frac{1}{q}} \\
\times \left( \int_0^1 \left| (\lambda + \nu)t - \lambda \right| \left| f'(tb + (1 - t)a) \right|^q \, dt \right)^{\frac{1}{q}}. \tag{19}
\]

Note that

\[
\int_0^1 \left| (\lambda + \nu)t - \lambda \right| \, dt = \frac{\lambda^2 + \nu^2}{2(\lambda + \nu)}, \tag{20}
\]

and

\[
\int_0^1 \left| (\lambda + \nu)t - \lambda \right| \left| f'(tb + (1 - t)a) \right|^q \, dt \\
= \int_0^\frac{\lambda}{\lambda + \nu} \left\{ \lambda - (\lambda + \nu)t \right\} \left| f'(tb + (1 - t)a) \right|^q \, dt \\
+ \int_{\frac{\lambda}{\lambda + \nu}}^1 \left\{ (\lambda + \nu)t - \lambda \right\} \left| f'(tb + (1 - t)a) \right|^q \, dt.
\]
Since $|f'|^q$ is geometrically convex and monotonically decreasing on $[a, b]$ for $q \geq 1$, we have

\[
(i) \int_0^{\frac{\lambda}{\lambda+\nu}} \left\{ \lambda - (\lambda + \nu)t \right\} \left| f'(tb + (1 - t)a) \right|^q \, dt \\
\leq \int_0^{\frac{\lambda}{\lambda+\nu}} \left\{ \lambda - (\lambda + \nu)t \right\} \left\{ \left| f'(a) \right|^{1-t} \left| f'(b) \right|^t \right\}^q \, dt \\
= \left| f'(a) \right|^q \left\{ \lambda \int_0^{\frac{\lambda}{\lambda+\nu}} M^q \, dt - (\lambda + \nu) \int_0^{\frac{\lambda}{\lambda+\nu}} t M^q \, dt \right\} \\
= \left| f'(a) \right|^q \left\{ \lambda \mu_3 (M, \frac{\lambda}{\lambda+\nu}, 0) - (\lambda + \nu) \mu_3 (M, 0, \frac{\lambda}{\lambda+\nu}) \right\}, \quad (21)
\]

(ii) \[
\int_0^1 \left\{ (\lambda + \nu)t - \lambda \right\} \left| f'(tb + (1 - t)a) \right|^q \, dt \\
\leq \left| f'(a) \right|^q \left\{ (\lambda + \nu) \mu_3 (M, \frac{\lambda}{\lambda+\nu}, 1) - \lambda \mu_3 (M, 1, \frac{\lambda}{\lambda+\nu}) \right\}. \quad (22)
\]

By substituting (20)-(22) in (19), we get the desired result.

**Theorem 3.2.** Let $f : I \to R$ be a differentiable function on the interior $I^0$ of an interval $I$ in $R_+$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is geometrically convex and monotonically decreasing on $[a, b]$, for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for $\lambda, \nu \in [0, \infty)$ with $\lambda + \nu > 0$ the following inequality holds:

\[
\left| \frac{\lambda f(a) + \nu f(b)}{\lambda + \nu} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\
\leq \left| f'(a) \right| \left( \frac{b-a}{\lambda + \nu} \right) \left( \frac{\lambda^{1+p} + \nu^{1+p}}{(1+p)(\lambda + \nu)} \right)^{\frac{1}{p}} \left( \frac{M^q - 1}{q \ln M} \right)^{\frac{1}{q}}.
\]

**Proof.** From Lemma 2 and using the Hölder integral inequality, we have

\[
\left| \frac{\lambda f(a) + \nu f(b)}{\lambda + \nu} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b-a}{\lambda + \nu} \int_0^1 \left| (\lambda + \nu)t - \lambda \right| \left| f'(tb + (1 - t)a) \right| \, dt \\
\leq \frac{b-a}{\lambda + \nu} \left( \int_0^1 \left| (\lambda + \nu)t - \lambda \right|^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'(tb + (1 - t)a) \right|^q \, dt \right)^{\frac{1}{q}}. \quad (23)
\]

Note that

\[
\int_0^1 \left| (\lambda + \nu)t - \lambda \right|^p \, dt = \frac{\lambda^{1+p} + \nu^{1+p}}{(1+p)(\lambda + \nu)} \quad (24)
\]
and, since $|f'|^q$ is geometrically convex and monotonically decreasing on $[a, b]$ for $q \geq 1$, we have

$$\int_0^1 \left| f'(tb + (1 - t)a) \right|^q dt \leq \int_0^1 \left( \left| f'(a) \right|^{1-t} \left| f'(b) \right|^t \right)^q dt \leq \left| f'(a) \right|^q M^q - 1 \frac{q}{q \ln M} .$$

(25)

By substituting (24) and (25) in (23), we get the desired result.

References


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