Coherent Risk Measures by Pricing Functionals

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Abstract
The aim of this paper is to introduce a new class of Coherent Risk Measures in Banach lattices, whose dual representation is consistent to the pricing functionals of an incomplete market. As a sub-class, we introduce the corresponding Expected Shortfall Risk Measures.

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1 Introduction
We consider two periods of time (0 and 1) and a non-empty set of states of the world $\Omega$ which is supposed to be an infinite set. The true state $\omega \in \Omega$ that the investors face is contained in some $A \in \mathcal{F}$, where $\mathcal{F}$ is some $\sigma$-algebra of subsets of $\Omega$ which gives the information about the states that may occur at time-period 1. A financial position is a $\mathcal{F}$-measurable random variable $x : \Omega \to \mathbb{R}$. This random variable is the profile of this position at time-period 1. We suppose that the probability of any state of the world to occur is given by a probability measure $\mu : \mathcal{F} \to [0, 1]$. The financial positions are supposed to lie in some subspace $E$ of $L^0(\Omega, \mathcal{F}, \mu)$, being a Banach lattice.

Definition 1.1 An incomplete market in $E$ is some sublattice $M$ of $E$. A complete market in $E$ is some sublattice $M$ of $E$, such that $\overline{M} = E$.

It is well-known that we define the positive cone $F_+$ of a subspace $F$ of an ordered vector space to be the set $F_+ = F \cap E_+$, where $E_+$ denotes the positive cone of $E$. 
Definition 1.2 A positive projection $P : E \to F$ is a projection, which maps each element of $E$ to some element of its subspace $F$, such that $P(E_+) \subseteq F_+$. A positive projection is called strictly positive, if $P(x) \in F_+, P(x) = 0 \iff x \in E_+, x = 0$.

Definition 1.3 A filtration associated to the pair $(T, E)$, where $T$ is a topological space, is a net of projections $(P_a)_{a \in A}$, where $P_a : E \to E_a$, where $E_a$ is a sublattice of $E$ and if $b \succeq a, P_a P_b = P_a$. $A$ is a directed set, by some binary relation $\succeq$, called direction.

Definition 1.4 A binary relation $\succeq$ on $A$ is called direction on $A$, if it is reflexive and transitive on $A$, while for any $a, b \in A$ there is a $c \in A$, such that $c \succeq a, b$.

In [6], we gave Examples of filtrations and we proved the following order versions of the Fundamental Theorems of Asset Pricing:

Theorem 1.5 (Order 1st Fundamental Theorem of Asset Pricing) Let $E$ be a Banach lattice and $M$ be a sublattice of $E$. If $M$ admits a strictly positive projection, then every strictly positive and continuous functional $f : M \to \mathbb{R}$, admits a strictly positive, continuous extension on $E$. Also, if $E$ is a Banach lattice and $M$ is a sublattice of $E$ such that every strictly positive and continuous functional $f : M \to \mathbb{R}$, admits a strictly positive, continuous extension on $E$, then $M$ admits a strictly positive projection.

Theorem 1.6 (Order 2nd Fundamental Theorem of Asset Pricing) Let $E$ be a Banach lattice and $M$ be a dense sublattice of $E$. If $M$ admits a strictly positive projection, then every strictly positive and continuous functional $f : M \to \mathbb{R}$, admits a unique strictly positive, continuous extension on $E$. Also, let $E$ be a Banach lattice and $M$ be a sublattice of $E$ such that $M$ admits a strictly positive projection. Moreover, every strictly positive and continuous functional $f : M \to \mathbb{R}$, admits a unique strictly positive, continuous extension on $E$. Then $M$ is dense in $E$.

These versions of FTAP provide the opportunity for the definition of a new class of Coherent Risk Measures, relying on the functionals being the strictly positive extensions of the equivalent functionals defined on the asset span $M$, instead of Radon-Nikodym derivatives of $\mu$-continuous probability measures. The idea of definition of a risk measure by using a general asset span $M$ is not new. It was first proposed in [4, p.183], in which no topological frame was used. Here, we use the specific versions of the FTAP in order to associate the pricing functionals of a market and the dual representation of a risk measure. Results in this direction in the infinite-dimensional case is the scope of this
paper, because this idea is contained as a seminal one in the finite-dimensional case, in [2, Cond.4.3, p.222]. The definition of the Expected Shortfall relies on the dual representation [5, Th.4.1] in $L^1$, which is generalized here for a variety of classes of Banach lattices.

2 Pricing Functionals and Coherent Risk Measures

Let $E$ be a Banach lattice and $M$ be a sublattice of $E$, which admits a strictly positive projection $P$. Then we define the non-empty set

$$D_M = \{ f \in E^* | f = P^*(g), g : M \to \mathbb{R} \},$$

where $g$ is strictly positive functional of $M_+ = E_+ \cap M$ and $f$ is a strictly positive functional of $E_+$. The fact that $D_M$ is well-defined and non-empty arises from Theorem 1.5. If $M$ corresponds to a complete market $D_M$ is a singleton, according to Theorem 1.6.

2.1 The case of an incomplete market

Lemma 2.1 Let $E$ be a Banach lattice and $M$ be a sublattice of $E$, which admits a strictly positive projection $P$. For any $u \in E_+, u \neq 0$ such that $u \notin \overline{M}$, and any $f \in D_M$, $f(-u) = 1$.

Proof: $\overline{M}$ is a convex, closed subset of $E$, while $\{u\}$ is a convex, compact subset of $E$. Hence, there is some $f \in E^*, f \neq 0$ such that

$$f(x) \geq a + \epsilon > a \geq f(u),$$

from the Convex Sets’ Separation Theorem, where $a \in \mathbb{R}, \epsilon > 0$ and $x \in \overline{M}$. Since $0 \in \overline{M}$, then the separation inequalities take the form $f(x) \geq 0 > f(u)$. Hence this $f$ is in general a positive functional of $M_+$, for any of which we may set $f(u) = -1$. This implies that for any $f \in D_M$, we take $f(u) = -1$.

Definition 2.2 A coherent risk measure associated with the pair $(E_+, u)$ (or an $(E_+, u)$-coherent risk measure, where $u \in E_+$) is a real-valued functional $\rho : E \to \mathbb{R}$ satisfying the properties:

(i) $\rho(x + tu) = \rho(x) - t$ for any $x \in E, t \in \mathbb{R}$ (u-Translation Invariance)

(ii) $\rho(x + y) \leq \rho(x) + \rho(y)$ for any $x, y \in E$ (Subadditivity)

(iii) $\rho(\lambda x) = \lambda \rho(x)$ for any $x \in E, \lambda \in \mathbb{R}_+$ (Positive Homogeneity)
(iv) $x \geq y$ in terms of the partial ordering $\geq$ of $E$ induced by $E_+$ implies $\rho(y) \geq \rho(x)$, for any $x, y \in E$ (Monotonicity).

**Theorem 2.3** Let $E$ be a Banach lattice and $M$ be a sublattice of $E$, which admits a strictly positive projection $P$. Then the functional

$$\rho_M : E \to \mathbb{R}, \rho(x) = \sup_{f \in D_M} f(-x),$$

is a $(E_+, u)$-coherent risk measure on $E$, where $u \in E_+$ is one of the vectors indicated by Lemma 2.1.

**Proof:** We can verify that $\rho_M$ satisfies the $u$-Translation Invariance, the Subadditivity, the Positive Homogeneity and the $E_+$-Monotonicity.

**Corollary 2.4** Let $E$ be an AM-space with order unit $u$ and $M$ be a sublattice of $E$, which admits a strictly positive projection $P$. Then the functional

$$\rho_M : E \to \mathbb{R}, \rho(x) = \sup_{f \in D_M} f(-x),$$

is a $(E_+, u)$-coherent risk measure on $E$, where $u \in E_+$ is the order unit $u$.

**Corollary 2.5** Let $E$ be a Banach lattice whose positive cone $E_+$ contains quasi-interior points and $M$ be a sublattice of $E$, which admits a strictly positive projection $P$. Then the functional

$$\rho_M : E \to \mathbb{R}, \rho(x) = \sup_{f \in D_M} f(-x),$$

is a $(E_+, u)$-coherent risk measure on $E$, where $u \in E_+$ is a quasi-interior point of $E_+$.

**Theorem 2.6** Let $E$ be the space $L^2$ over the probability space $(\Omega, \mathcal{F}, \mu)$ and $M$ be a sublattice of $E$, which admits a strictly positive projection $P$. Then the functional

$$ES_{a,M,u} : E \to \mathbb{R}, ES_{a,M,u}(x) = \sup_{f \in D_M \cap [0, \frac{1}{u}]} f(-x),$$

is a $(E_+, u)$-coherent risk measure on $E$, where $u \in E_+$ is one of the vectors indicated by Lemma 2.1.

**Proof:** We can verify that $\rho_M$ satisfies the $u$-Translation Invariance, the Subadditivity, the Positive Homogeneity and the $E_+$-Monotonicity.
Corollary 2.7 Let $E$ be an AL-space, whose dual space $E^*$ has an order unit $e$. Also, let $M$ be a sublattice of $E$, which admits a strictly positive projection $P$. Then the functional
\[ E_{a,M,u,e} : E \to \mathbb{R}, \quad E_{a,M,u,e}(x) = \sup_{f \in D_M \cap [0,\frac{1}{a} e]} f(-x), \]
is a $(E_+, u)$-coherent risk measure on $E$, where $u \in E_+$ is one of the vectors indicated by Lemma 2.1.

Corollary 2.8 Let $E$ be a Banach lattice, whose dual space positive cone has quasi-interior points, while $e \in E_+^*$ is one of them. Also, let $M$ be a sublattice of $E$, which admits a strictly positive projection $P$. Then the functional
\[ E_{a,M,u,e} : E \to \mathbb{R}, \quad E_{a,M,u,e}(x) = \sup_{f \in D_M \cap [0,\frac{1}{a} e]} f(-x), \]
is a $(E_+, u)$-coherent risk measure on $E$, where $u \in E_+$ is one of the vectors indicated by Lemma 2.1.

Definition 2.9 Each of the risk measures $E_{a,M,u}$ is called Expected Shortfall with respect to $(a, M, u)$, where $a \in (0, 1)$ denotes a level of significance, $M$ denotes the market space and $u$ denotes the insurance asset.

2.2 The case of a complete market

Theorem 2.10 Let $E$ be a Banach lattice and $M$ be a dense sublattice of $E$, which admits a strictly positive projection $P$. Then the functional
\[ \rho_M : E \to \mathbb{R}, \quad \rho(x) = f_0(-x), \]
is a $(E_+, u)$-coherent risk measure on $E$, where $u \in E_+, u \neq 0$ is such $f_0(u) = 1$, while $D_M = \{ f_0 \}$ in this case.

Proof: We can verify that $\rho_M$ satisfies the u-Translation Invariance, the Subadditivity, the Positive Homogeneity and the $E_+$-Monotonicity. Since $f_0$ is a strictly positive functional of $E_+$ and $u \in E_+, u \neq 0$, $f_0(u) > 0$, which may be set to be equal to 1.

Theorem 2.11 Let $E$ be a Banach lattice whose dual space positive cone contains quasi-interior points, while $e \in E_+^*$ is one of them. Let $M$ be a dense sublattice of $E$, which admits a strictly positive projection $P$. Then the functional
\[ E_{a,M,u,e} : E \to \mathbb{R}, \quad E_{a,M}(x) = f_0(-x), \]
is a $(E_+, u)$-coherent risk measure on $E$, where $u \in E_+, u \neq 0$ is such $f_0(u) = 1$, while $D_M = \{ f_0 \}$ in this case. $u \in E_+$ is such $f_0 \in [0,\frac{1}{a} e]$, while $D_M = \{ f_0 \}$ in this case.
Proof: The same as above.

Corollary 2.12 Let $E$ be an AL-space whose dual space $E^*$ has an order unit $e \in E^*_+$. Let $M$ be a dense sublattice of $E$, which admits a strictly positive projection $P$. Then the functional

$$ES_{a,M,u,e} : E \rightarrow \mathbb{R}, \quad ES_{a,M}(x) = f_0(-x),$$

is a $(E_+,u)$-coherent risk measure on $E$, where $u \in E_+, u \neq 0$ is such $f_0(u) = 1$, while $D_M = \{ f_0 \}$ in this case. $u \in E_+$ is such $f_0 \in [0, \frac{1}{a}e]$, while $D_M = \{ f_0 \}$ in this case.

Proof: The same as above.

3 Appendix

In this paragraph, we give some essential notions and results from the theory of partially ordered linear spaces which are used in the previous sections of this article.

Let $L$ be a (normed) linear space. A set $C \subseteq L$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \in \mathbb{R}_+$ is called wedge. A wedge for which $C \cap (-C) = \{0\}$ is called cone. A pair $(L, \geq)$ where $L$ is a linear space and $\geq$ is a binary relation on $L$ satisfying the following properties:

(i) $x \geq x$ for any $x \in L$ (reflexive)

(ii) If $x \geq y$ and $y \geq z$ then $x \geq z$, where $x, y, z \in L$ (transitive)

(iii) If $x \geq y$ then $\lambda x \geq \lambda y$ for any $\lambda \in \mathbb{R}_+$ and $x + z \geq y + z$ for any $z \in L$, where $x, y \in L$ (compatible with the linear structure of $L$),

is called partially ordered linear space.

The binary relation $\geq$ in this case is a partial ordering on $L$. The set $P = \{ x \in L | x \geq 0 \}$ is called (positive) wedge of the partial ordering $\geq$ of $L$. Given a wedge $C$ in $L$, the equivalent partial ordering relation $\geq_C$ defined as follows:

$$x \geq_C y \iff x - y \in C,$$

is a partial ordering on $L$, called partial ordering induced by $C$ on $L$. If the partial ordering $\geq$ of the space $L$ is antisymmetric, namely if $x \geq y$ and $y \geq x$ implies $x = y$, where $x, y \in L$, then $P$ is a cone. If $L$ is partially ordered by $C$, then any set of the form $[x, y] = \{ r \in L | y \geq_C r \geq_C x \}$ where $x, y \in C$ is called order-interval of $L$. $e$ is an order unit of $L$, if $L = \bigcup_{n=1}^{\infty}[-ne, ne] = I_e$, where $[-ne, ne]$ is the order-interval of $L$ and we suppose that $L$ is ordered by
$L_+$, while $n \in \mathbb{N}$. If $e$ is an interior point, then $e$ is an order unit of $L$. $e$ is a quasi-interior point of $L$ if and only if $L = \overline{Le}$.

If $e$ is an order unit and $L$ is a Banach space, then $e$ is an interior point of $L$. $L'$ denotes the linear space of all linear functionals of $L$, while $L^*$ is the norm dual of $L^*$, in case where $L$ is a normed linear space.

Suppose that $C$ is a wedge of $L$. A functional $f \in L'$ is called positive functional of $C$ if $f(x) \geq 0$ for any $x \in C$. $f \in L'$ is a strictly positive functional of $C$ if $f(x) > 0$ for any $x \in C \setminus \{0\}$.

The partially ordered vector space $L$ whose positive cone is $P$ is a vector lattice if for any $x, y \in L$, the supremum and the infimum of $\{x, y\}$ with respect to the partial ordering defined by the cone $P$ exist in $L$. In this case $\sup \{x, y\}$ and $\inf \{x, y\}$ are denoted by $x \vee y, x \wedge y$ respectively. If so, $|x| = \sup \{x, -x\}$ is the absolute value of $x$ and if $L$ is also a normed space such that $\|x\| = \|x\|$ for any $x \in L$, then $E$ is called normed lattice. If $L$ is also a Banach space, then $L$ is a Banach lattice. If $L$ is a Banach lattice, whose norm has the property: $\|x + y\| = \|x\| + \|y\|, x, y \in L_+$, then $L$ is called $AL$-space. If $L$ is a Banach lattice, whose norm has the property: $\|x \vee y\| = \max\{\|x\|, \|y\|\}, x, y \in L_+$, then $L$ is called $AM$-space.

All the previously mentioned notions and related propositions concerning partially ordered linear spaces are contained in [3] and [1].

References


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