Abstract

Some Newton-Cote’s type of quadrature rules have been formulated for the numerical evaluation of hyper singular integrals, which are interpreted as of Hadamard finite part type. Rules have been tested numerically by some standard test integrals and their respective error bounds have been determined.

Mathematics Subject Classification: 65D30; 65D32

Keywords: Singularity; Hilbert transform; Cauchy principal value; Hadamard finite part integral; residue; asymptotic error estimate; error bound
1 Introduction

The finite Hilbert transform [9] of the type:

\[ I(f, c) = P \int_{a}^{b} \frac{f(x)}{x - c} \, dx; \quad a < c < b \] (1)

where \( f(x) \) is sufficiently differentiable in \([a, b]\) lead to uncontrolled instability when standard quadrature rules (either Newton-Cote’s type or Gauss type) are to be applied for their approximate evaluation, due to the presence of the singularity \( x = c \) in the range of integration.

The Cauchy principal value of this integral is defined as:

\[ I(f, c) = \lim_{\varepsilon \to 0^+} \left[ \int_{a}^{c-\varepsilon} \frac{f(x)}{x - c} \, dx + \int_{c+\varepsilon}^{b} \frac{f(x)}{x - c} \, dx \right] \]

provided the limit exists. In case this limit exists, the limiting value is known as the Cauchy-principal value (CPV) and the integral is denoted as by equation(1).

Singular integrals of the type (1) occur frequently in many branches of physics, in the theory of aerodynamics and scattering theory etc. However, the quadrature rules meant for the numerical integration of the integral (1) behave very much unstable when these rules are applied for the approximation of the integral:

\[ J(f, c) = \int_{a}^{b} \frac{f(x)}{(x - c)^{\alpha}} \, dx, \quad \alpha > 1; \quad a < c < b \] (2)

due to the presence of higher order singularity at \( x = c \). Further, the divergent integral:

\[ \int_{a}^{b} \frac{f(x)}{(x - c)^{2}} \, dx = \lim_{\varepsilon \to 0^+} \left[ \int_{a}^{c-\varepsilon} \frac{f(x)}{(x - c)^{2}} \, dx + \int_{c+\varepsilon}^{b} \frac{f(x)}{(x - c)^{2}} \, dx \right] ; \] (3)

is very much expressed as:

\[ \int_{a}^{b} \frac{f(x)}{(x - c)^{2}} \, dx = H \int_{a}^{b} \frac{f(x)}{(x - c)^{2}} \, dx + \lim_{\varepsilon \to 0^+} \frac{2f(c)}{\varepsilon} . \] (4)

The integral present in the right side of the above equation(4) is called as the Hadamard finite-part integral[9]. Ramm and Van der Sluis [1], Groetsch[2], Criscuolo[6], Paget[4], Elliott[3] and many more as available in literature have been contributed their work for the approximate evaluation of this integral. However the basic purpose of this paper is to formulate some quadrature rules which are uniformly convergent to the Cauchy principal value of the integral of the type (1); and the same rule has been employed for the approximate evaluation of the finite part integral (2) by reducing the order of singularity. For this, we consider the integrals of the type equation(1) and the integrals of Hadamard type as given in equation(2), for \( \alpha = 2 \) in the interval\([-a, a]\).
2. Formulation of Quadrature Rules

This section has two subsections: subsection-2.1 and subsection-2.2 as given below.

2.1 Rules for the Approximate Evaluation of Real CPV Integrals

The (4n−1)-point rule is generated by decomposing the interval of integration \([-a, a]\) into \((4n - 2)\) equal parts by the points:

\[
0, \pm \frac{a}{2n}, \pm \frac{2a}{2n}, \pm \frac{3a}{2n}, \ldots, \pm \frac{(2n-1)a}{2n},
\]

is denoted by \(R_n(f)\) and defined as:

\[
R_n(f) = w_{n0}f(0) + \sum_{k=1}^{(2n-1)} w_{nk} \left[ f \left( \frac{ka}{2n} \right) - f \left( \frac{ka}{2n} \right) \right]
\]

(5)

Since the nodes are prefixed, thus it is only remain to determine the coefficients \(w_{n0}\) and \(w_{nk}\); for \(k = 1(1)(2n - 1)\) associated with \(f(0)\) and with the block \(|f(\frac{ka}{2n}) - f(\frac{ka}{2n})|\) respectively.

It is to be noted here that in such rules the coefficient of \(f(0)\) i.e. \(w_{n0}\) is zero, for all \(n\). The coefficients \(w_{nk}\) of the rule \(R_n(f)\) for all \(n\) associated with the above block are the solutions of the following set of moment equations \(AW = B\) in coefficients \(w_{nk}\); where

\[
A = \begin{pmatrix}
1 & 2 & \cdots & (2n-1) \\
1^3 & 2^3 & \cdots & (2n-1)^3 \\
1^5 & 2^5 & \cdots & (2n-1)^5 \\
\vdots & \vdots & \ddots & \vdots \\
1^{2n-1} & 2^{2n-1} & \cdots & (2n-1)^{2n-1}
\end{pmatrix},
\]

\[W = \begin{pmatrix}
w_{n1} \\
w_{n2} \\
\vdots \\
w_{n(2n-1)}
\end{pmatrix},
\]

\[B = \begin{pmatrix}
\frac{2n}{1} \\
\frac{(2n)^3}{3} \\
\vdots \\
\frac{(2n)^{2n-1}}{2n-1}
\end{pmatrix}.
\]

The rules corresponding to \(n = 1, 2, 3\) and 4 are noted below.

\[R_1(f) = 2 \left[ f \left( \frac{a}{2} \right) - f \left( -\frac{a}{2} \right) \right].
\]

\[R_2(f) = \frac{134}{45} \left[ f \left( \frac{a}{4} \right) - f \left( -\frac{a}{4} \right) \right] - \frac{206}{225} \left[ f \left( \frac{a}{2} \right) - f \left( -\frac{a}{2} \right) \right] - \frac{214}{225} \left[ f \left( \frac{3a}{4} \right) - f \left( -\frac{3a}{4} \right) \right].
\]

\[R_3(f) = \frac{4433}{255} \left[ f \left( \frac{a}{6} \right) - f \left( -\frac{a}{6} \right) \right] - \frac{2717}{337} \left[ f \left( \frac{a}{3} \right) - f \left( -\frac{a}{3} \right) \right] + \frac{14779}{2450} \left[ f \left( \frac{a}{2} \right) - f \left( -\frac{a}{2} \right) \right] - \frac{422}{201} \left[ f \left( \frac{2a}{3} \right) - f \left( \frac{2a}{3} \right) \right] + \frac{489}{614} \left[ f \left( \frac{5a}{6} \right) - f \left( -\frac{5a}{6} \right) \right]; \text{ and}
\]

\[R_4(f) = \frac{14687}{289} \left[ f \left( \frac{a}{8} \right) - f \left( -\frac{a}{8} \right) \right] - \frac{4316}{63} \left[ f \left( \frac{a}{4} \right) - f \left( -\frac{a}{4} \right) \right] + \frac{4044}{71} \left[ f \left( \frac{3a}{8} \right) - f \left( -\frac{3a}{8} \right) \right] - \frac{4376}{139} \left[ f \left( \frac{a}{2} \right) - f \left( -\frac{a}{2} \right) \right] + \frac{2927}{232} \left[ f \left( \frac{5a}{8} \right) - f \left( -\frac{5a}{8} \right) \right] - \frac{1124}{357} \left[ f \left( \frac{3a}{4} \right) - f \left( -\frac{3a}{4} \right) \right] + \frac{485}{671} \left[ f \left( \frac{7a}{8} \right) - f \left( -\frac{7a}{8} \right) \right].
\]

It is pertinent to note here that for all \(n\), the rule \(R_n(f)\) is an open type rule since both the end points \(-a\) and \(a\) of the interval of integration \([-a, a]\) are excluded in the set of \((4n - 1)\) nodes.

2.2. Scheme for the Numerical Approximation of Hadamard Finite Part Integrals of the Type

\[J(f) = H \int_{-a}^{a} \frac{f(x)}{x^2} \, dx.
\]

(10)

For the construction of the scheme, here we assume that \(f(z)\) is the analytic continuation of \(f(x)\) in the disc:

\[\Omega = \{z \in \mathbb{C}: |z| \leq r: r > a\}.
\]
As a result, \( f(z) = f(x); \forall x \in [-a, a]. \)

The integral given in (10) can be transformed to:

\[
J(f) = \int_{-a}^{a} \frac{f(x)}{x^2} dx = K + L; \quad \text{where}
\]

\[
K = \int_{-a}^{a} \left( \frac{f(x) - K_1}{x} \right) dx; \quad K_1 = \frac{x}{a} \int_{-a}^{a} f(x) \, dx
\]

\[
= \int_{-a}^{a} \frac{g(x)}{x} dx; \quad g(x) = \left( \frac{f(x) - K_1}{x} \right)
\]

\[
\approx R_n(g); \quad \text{and} \quad L = \int_{-a}^{a} \frac{K_1}{x^2} dx.
\]

Now since the integral \( K \) is a singular integral of the type (1), thus the rules \( R_1(f) \) to \( R_4(f) \) as given from equations (6) to (9) meant for the numerical integration of the real CPV integrals may be applied for its numerical approximations. Further, though the integral \( L \) is also a singular integral but as its primitive \((-\frac{1}{x})\) exists, thus it can be evaluated by using the Fundamental Theorem of Integral Calculus and thus

\[
L = \int_{-a}^{a} \frac{K_1}{x^2} dx = -\frac{2K_1}{a}.
\]

As a result the integral:

\[
J(f) = \int_{-a}^{a} \frac{f(x)}{x^2} dx \approx R_n(g) - \frac{2K_1}{a}; \quad \text{where}
\]

\[
g(x) = \left( \frac{f(x) - K_1}{x} \right)
\]

and \( a \) is the one of the end points of the interval of integration \( [-a, a] \). Next we consider:

3. Error Analysis

The error bounds of the truncation error \( E_n(f) \) associated with the quadrature rules \( R_n(f) \) for \( n = 1, 2, 3 \) and 4 for the numerical evaluations of real Cauchy principal value of integrals (Equations (6) to (9)) have been determined by following the techniques due to Lether[5] and is given in Theorem-1. Since derivation of each parts of the Theorem-1 are similar we have derived only the part-(i) to avoid repetition.

**Theorem-1.** If \( f(z) \) is an analytic continuation of \( f(x) \) defined in the closed disc: \( \Omega = \{ z \in C : |z| \leq r; r > a \} \); then

(i) \( |E_1(f)| \leq Me_{1a}(r) \);

(ii) \( |E_1(f)| \leq Me_{2a}(r) \);

(iii) \( |E_3(f)| \leq Me_{3a}(r) \);

(iv) \( |E_4(f)| \leq Me_{4a}(r) \);

where \( M = \max_{|z| \leq r} |f(z)| \); \( e_{1a}(r) = |\ln \left( \frac{z+1}{z-1} \right) - \frac{4r}{\pi z^2} \}; \)

\[
e_{2a}(r) = |\ln \left( \frac{z+1}{z-1} \right) - \frac{42r}{225 \cdot (6a^2-1)} - \frac{277r}{15 \cdot (6a^2-1)} + \frac{147r}{25 \cdot (6a^2-1)} + \frac{89r}{30 \cdot (6a^2-1)}|;
\]

\[
e_{3a}(r) = |\ln \left( \frac{z+1}{z-1} \right) - \frac{4433r}{25 \cdot (6a^2-1)} - \frac{527r}{5 \cdot (6a^2-1)} + \frac{121r}{4 \cdot (6a^2-1)} + \frac{14779r}{4 \cdot (6a^2-1)} + \frac{489r}{30 \cdot (6a^2-1)}|;
\]

\[
e_{4a}(r) = |\ln \left( \frac{z+1}{z-1} \right) - \frac{4687r}{289 \cdot (6a^2-1)} - \frac{4316r}{63 \cdot (6a^2-1)} + \frac{4044r}{71 \cdot (6a^2-1)} - \frac{4376r}{139 \cdot (6a^2-1)} + \frac{4433r}{25 \cdot (6a^2-1)}|.
\]

Each of which \( \to 0 \) as \( r \to \infty \). The quantity \( e_{ka}(r) \) is defined as error constant due to Lether[5].

**Proof-(i):** Let

\[
f(z) = f(x); \quad \forall z \in [-a, a].
\]

Now by expanding \( f(z) \) by Taylor’s theorem about \( z = 0 \) we have

\[
f(z) = \sum_{k=0}^{\infty} c_k z^k
\]

where \( c_k = \frac{f^{(k)}(0)}{k!} \); for \( k = 0, 1, 2, \ldots, z \in [-a, a] \) are the Taylor’s coefficients.

Now since \( E_1 \) being a linear operator, we obtain

\[
|E_1(f)| \leq \sum_{k=1}^{\infty} |c_{2\mu+1}| |E_1(x^{2\mu+1})|.
\]
### Table 1: Values of Error Constants $e_{na}(r)$ for $r > 1$ and $a = 1$

<table>
<thead>
<tr>
<th>$r$</th>
<th>$e_{1a}(r)$</th>
<th>$e_{2a}(r)$</th>
<th>$e_{3a}(r)$</th>
<th>$e_{4a}(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>0.7528557711</td>
<td>0.2426583654</td>
<td>0.0842135785</td>
<td>0.0312700405</td>
</tr>
<tr>
<td>1.8</td>
<td>0.048745906</td>
<td>0.013995264</td>
<td>0.000371917</td>
<td>0.000010571</td>
</tr>
<tr>
<td>2.7</td>
<td>0.0106591140</td>
<td>0.000567571</td>
<td>0.00002612</td>
<td>0.00000013</td>
</tr>
<tr>
<td>3.6</td>
<td>0.0040617743</td>
<td>0.000067383</td>
<td>0.00000994</td>
<td>0.00000000</td>
</tr>
</tbody>
</table>

### Table 2: Asymptotic Error Bounds of $R_n(f)$ for $n = 1, 2, 3$ and $4$

<table>
<thead>
<tr>
<th>Rules</th>
<th>First Leading Term of Error Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1(f)$</td>
<td>$0.03 \times f^{(3)}(0)$</td>
</tr>
<tr>
<td>$R_2(f)$</td>
<td>$9.8 \times 10^{-6} f^{(7)}(0)$</td>
</tr>
<tr>
<td>$R_3(f)$</td>
<td>$2.6 \times 10^{-10} f^{(11)}(0)$</td>
</tr>
<tr>
<td>$R_4(f)$</td>
<td>$1.8 \times 10^{-15} f^{(15)}(0)$</td>
</tr>
</tbody>
</table>

Thus, by Cauchy Inequality \[8\]

$$|E_1(f)| \leq M \sum_{\mu=1}^{\infty} \frac{1}{r^{2\mu+1}} |E_1(x^{2\mu+1})|. \quad (13)$$

and then by using the technique due to Lether \[5\] we obtain

$$|E_1(f)| \leq M e_{1a}(r); \text{where}$$

$$e_{1a}(r) = E_1 \left[ \left( 1 - \frac{x}{r} \right)^{-1} \right] = \ln \left( \frac{r+1}{r-1} \right) - \frac{8r}{4r^2 - 1}$$

which $\to 0$ as $r \to \infty$. This proofs the part-(i) of Theorem-1 and hence the theorem is established.

### Comparative Study of the Error Constants

For $a = 1$, the error constants $e_{na}(r)$ corresponding to the rules $R_n(f)$ for $n = 1, 2, 3$ and $4$ have been evaluated for values of $r > 1$ and the results of computation are given in Table-1. It is observed from the Table-1 of values of error constants and the corresponding graphs(Fig.-1)drawn based on the table that

$$e_{4a}(r) < e_{3a}(r) < e_{2a}(r) < e_{1a}(r).$$

Also it is evident from Table-2 that the degree of precision of the rules are 2, 6, 10 and 14 respectively. In general the degree of precision of the rule $R_n(f)$ is $(4n - 2)$. Next we consider:

### 4. Numerical Experiments

This article consists of two parts: Part-I to Part-II.

**Part-I. Approximate Evaluation of Real CPV Integral with Singularity at Origin**

The integral considered here is:
Rules | Approx. of $I$ | Abs.Err
---|---|---
$R_1(f)$ | 2.084381221975 | 0.03
$R_2(f)$ | 2.114492444125 | $9.3 \times 10^{-6}$
$R_3(f)$ | 2.114501750492 | $2.6 \times 10^{-10}$
$R_4(f)$ | 2.114501750752 | 0.0

Table 3: Evaluation of Real CPV Integral with Singularity at Origin

$I = \int_{-1}^{1} \frac{x}{x^2} dx = 2.114501750752\ (\text{Ref. Longman[7]})$

and the result of numerical approximations are given in Table-3.

**Part-II. Approx.of Hadamard Finite Part Integral**

$$J = H \int_{-1}^{1} \frac{x}{x^2} dx = -0.971659518878$$

The result of numerical integrations of this integral is given in Table-4.

**5. Conclusion**

From the Tables of numerical results it is observed that the values obtained by the sequence of rules of increasing precision converges to a value i.e. equal to the exact value of the respective integrals correct up to at least ten figures after the decimal point in case of real CPV integrals as well as integrals of the type Hadamard finite part. Also, it is not require to evaluate derivative of the integrand at any of its nodes, which is a positive advantage over the existing rules found in the literature.

Rules | Approx. of $J$ | Abs.Err
---|---|---
$R_1(f)$ | -0.978992278349 | 0.01
$R_2(f)$ | -0.971660676385 | $1.2 \times 10^{-6}$
$R_3(f)$ | -0.971659518901 | $2.3 \times 10^{-11}$
$R_4(f)$ | -0.971659518878 | 0.0

Table 4: Experiments on Hadamard Finite Part Integral
References


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