On Semideterministic Finite Automata

Games Type

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Abstract

To describe a state flow in situational theory of conflict in this paper we propose a model that combines the theory of Markov chains and game theory as a finite-state machine. A complete classification of low-dimensional automata and some analytical expressions for the average Bayesian payoff are obtained.

Mathematics Subject Classification: 60J20, 90C40, 91-04, 93C85

Keywords: finite-state automata, game theory, Bayesian strategy

1 Introduction

The classical theory of finite-state automata [1] supposes its work with the deterministic meaning of the input signal. I.e. for any input symbol, there exists a unique state of the next transition. In other words, deterministic finite-state automata given as quintuple

\[ A = (s, x, y, f, g), \]

where \( s \) is a finite set of states; \((x, y)\) is the input and output alphabet, respectively; \( f \) is the state-transition function; \( g \) is the output function. A natural generalization of deterministic finite-state machine is to use of random variables as its arguments. For example, in [2] Rabin introduced nondeterministic
automata, in which the transition function \( f \) was defined by a stochastic matrix. In [3] a probabilistic finite-state automaton game type has been suggested. Input of this automaton is a sequence of player’s strategies \((a_0, a_1, ..., a_k)\) and \((b_0, b_1, ..., b_m)\)

\[
\begin{array}{cccccccc}
  & a_0b_0 & a_0b_1 & \ldots & a_0b_m & a_1b_0 & \ldots & a_kb_m \\
S_0 & S_* & S_* & \ldots & S_* & S_* & \ldots & S_* \\
S_1 & S_* & S_* & \ldots & S_* & S_* & \ldots & S_* \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
S_n & S_* & S_* & \ldots & S_* & S_* & \ldots & S_* \\
\end{array}
\]

and output is a payoff matrix of games for current state of the finite-state machine.

\[
\begin{array}{cccccccc}
  & a_0b_0 & a_0b_1 & \ldots & a_0b_m & a_1b_0 & \ldots & a_kb_m \\
S_0 & c^0_{00} & c^0_{01} & \ldots & c^0_{0m} & c^0_{10} & \ldots & c^0_{km} \\
S_1 & c^1_{00} & c^1_{01} & \ldots & c^1_{0m} & c^1_{10} & \ldots & c^1_{km} \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
S_n & c^n_{00} & c^n_{01} & \ldots & c^n_{0m} & c^n_{10} & \ldots & c^n_{km} \\
\end{array}
\]

The number of payment matrices is determined by the number of possible states of the automaton \((S_0, S_1, ..., S_n)\):

\[
p(S_0) = \begin{pmatrix} c^0_{00} & c^0_{01} & \ldots & c^0_{0m} \\
   c^1_{00} & c^1_{01} & \ldots & c^1_{0m} \\
   \vdots & \vdots & \ldots & \vdots \\
   c^n_{00} & c^n_{01} & \ldots & c^n_{0m} \\
\end{pmatrix}, \quad \ldots, \quad p(S_n) = \begin{pmatrix} c^n_{00} & c^n_{01} & \ldots & c^n_{0m} \\
   c^n_{10} & c^n_{11} & \ldots & c^n_{1m} \\
   \vdots & \vdots & \ldots & \vdots \\
   c^n_{k0} & c^n_{k1} & \ldots & c^n_{km} \\
\end{pmatrix}.
\]

Probabilistic characteristics of the machine arise in cases, when one from two players is the nature. Then the human response strategies can be calculated using, for example, the Bayesian criterion. In paper [3] with use of this machine was simulated concrete situation of crash of the flood dam during floods on the Amur River in 2013. In [4], this scheme was used for situational simulation of the Zeya hydroelectric power station in extreme situations. In those articles the system had 3 and 5 states, respectively. In those works the authors did not find an analytical solution of tasks. In this connection was used StateFlow simulation methods with help Simulink Matlab package. The aim of this work is the classification of all possible states of Bayesian machine type used in [3] (with 3 states)

\[
A = \left( (s_1, s_2, s_3), \begin{pmatrix} x_{11} & x_{12} \\
   x_{21} & x_{22} \end{pmatrix}, \begin{pmatrix} y_{11} & y_{12} \\
   y_{21} & y_{22} \end{pmatrix}, f, g \right).
\]

and analytical solution to some of them.
2 Preliminary Notes

Consider steps of analytical solution of Bayesian automata (1).

1. If output function is

<table>
<thead>
<tr>
<th></th>
<th>$a_0b_0$</th>
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<th>$a_1b_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>$c_{00}^0$</td>
<td>$c_{01}^0$</td>
<td>$c_{10}^0$</td>
<td>$c_{11}^0$</td>
</tr>
<tr>
<td>$S_1$</td>
<td>$c_{00}^1$</td>
<td>$c_{01}^1$</td>
<td>$c_{10}^1$</td>
<td>$c_{11}^1$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$c_{00}^2$</td>
<td>$c_{01}^2$</td>
<td>$c_{10}^2$</td>
<td>$c_{11}^2$</td>
</tr>
</tbody>
</table>

then payment matrix are:

$$p(s_0) = \begin{pmatrix} c_{00}^0 & c_{01}^0 \\ c_{10}^0 & c_{11}^0 \end{pmatrix}, \quad p(s_1) = \begin{pmatrix} c_{00}^1 & c_{01}^1 \\ c_{10}^1 & c_{11}^1 \end{pmatrix}, \quad p(s_2) = \begin{pmatrix} c_{00}^2 & c_{01}^2 \\ c_{10}^2 & c_{11}^2 \end{pmatrix}.$$

2) For a given probability of arrival of input signal $b_k$ from Bayesian criterion we find an optimal strategies for all games $a_k^*$. (Obviously, they do not necessarily coincide.)

3) The product $a_k^*b_m$ generates input (i.e. argument function $f$), which determine the change in a system state at the next step

<table>
<thead>
<tr>
<th></th>
<th>$a_0b_0$</th>
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</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>$S_{k1}$</td>
<td>$S_{k2}$</td>
<td>$S_{k3}$</td>
<td>$S_{k4}$</td>
</tr>
<tr>
<td>$S_1$</td>
<td>$S_{k5}$</td>
<td>$S_{k6}$</td>
<td>$S_{k7}$</td>
<td>$S_{k8}$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$S_{k9}$</td>
<td>$S_{k10}$</td>
<td>$S_{k11}$</td>
<td>$S_{k12}$</td>
</tr>
</tbody>
</table>

For simplicity, we will to use the following notation. Because our machine has 3 states, the set of all possible transition $f$ form a 3-ary logical function of $3 \cdot 2^2 = 12$ variables (i.e. total $N = 3^{12} = 531441$ automata). However, because of Nash’s theorem asserts that any finite game has a solution in pure or mixed strategies, the number of different Bayesian machine is reduced by half.

Further we will use for classification only zero strategies of nature, given that for other automata the solutions can be obtained by redefining of the number of strategy and components of the payoff matrix.

For example, function

<table>
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<th>$a_0b_0$</th>
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<th>$a_1b_0$</th>
<th>$a_1b_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>$S_1$</td>
<td>$S_1$</td>
<td>$S_1$</td>
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<tr>
<td>$S_2$</td>
<td>$S_0$</td>
<td>$S_1$</td>
<td>$S_2$</td>
<td>$S_1$</td>
</tr>
</tbody>
</table>
corresponds to the sequence \((S_1, S_1, S_2, S_2, S_0, S_1) = (112201)\). Stateflow diagram of this function has the form.

![Stateflow diagram](image)

Further, using the symmetry properties the remaining machines may be grouped as equivalent i.e. having the same limit states and average winnings. For example machines \((000,000.000001, 000002, ..., 002222)\) are equivalent. Similarly, the machines

\[
\{010212 \equiv 010221 \equiv 012012 \equiv 100212 \equiv 01221 \equiv 100221 \equiv 102012\}
\]

are equivalent. As a result, we obtain \(6^3 = 216\) different type of machines

\[
(i_1 i_2 j_1 j_2 k_1 k_2), \quad (2)
\]

where \(i_1, i_2, j_1, j_2, k_1, k_2 = 0, 1, 2; \ i_1 \leq i_2, \ j_1 \leq j_2, \ k_1 \leq k_2\).

Now we remove the automata in which the transition functions is irrelevant. For example, for automaton \((000022)\) the last four values are not significant, since the states \(S_1\) and \(S_2\) are unattainable by definition. Finally, amount of machines is reduced to 57:

<table>
<thead>
<tr>
<th>(i_1 i_2)</th>
<th>(j_1 j_2 k_1 k_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>01</td>
<td>0200 0201 0202 0211 0212 0222</td>
</tr>
<tr>
<td></td>
<td>1200 1201 1202 1211 1212 1222</td>
</tr>
<tr>
<td></td>
<td>2200 2201 2202 2211 2212 2222</td>
</tr>
<tr>
<td>11</td>
<td>0200 0201 0202 0211 0212 0222</td>
</tr>
<tr>
<td></td>
<td>1200 1201 1202 1211 1212 1222</td>
</tr>
<tr>
<td></td>
<td>2200 2201 2202 2211 2212 2222</td>
</tr>
<tr>
<td>12</td>
<td>0000 0001 0002 0011 0012 0022</td>
</tr>
<tr>
<td></td>
<td>0201 0101 0102 0111 0112 0122</td>
</tr>
<tr>
<td></td>
<td>0211 0212 0222 1111 1112 1122</td>
</tr>
<tr>
<td></td>
<td>2211 2212 1222</td>
</tr>
</tbody>
</table>
3 Analytical solution of 112222-automaton.

This machine has the following transition function

<table>
<thead>
<tr>
<th></th>
<th>(a_0b_0)</th>
<th>(a_0b_1)</th>
<th>(a_1b_0)</th>
<th>(a_1b_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_0)</td>
<td>(S_1)</td>
<td>(S_1)</td>
<td>(S_*)</td>
<td>(S_*)</td>
</tr>
<tr>
<td>(S_1)</td>
<td>(S_2)</td>
<td>(S_2)</td>
<td>(S_*)</td>
<td>(S_*)</td>
</tr>
<tr>
<td>(S_2)</td>
<td>(S_2)</td>
<td>(S_*)</td>
<td>(S_*)</td>
<td>(S_*)</td>
</tr>
</tbody>
</table>

and payoff matrix

\[
p(s_0) = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}, \quad p(s_1) = \begin{pmatrix} d_{00} & d_{01} \\ d_{10} & d_{11} \end{pmatrix}, \quad p(s_2) = \begin{pmatrix} e_{00} & e_{01} \\ e_{10} & e_{11} \end{pmatrix}.
\]

Suppose Bayes criterion suggests optimal choice of the zero-point strategies for all games. We denote these strategies by \(a^* = a_0\). Then for simplicity and without loss of generality, we relabel the entries of payoff matrix as

\[
c_0 = c_{00}, \quad c_1 = c_{01}, \quad d_0 = d_{00}, \quad d_1 = d_{01}, \quad e_0 = e_{00}, \quad e_1 = e_{01}.
\]

We now introduce the notation

\[
\mu_0 = b_0c_0 + b_1c_1, \quad \mu_1 = b_0d_0 + b_1d_1, \quad \mu_2 = b_0e_0 + b_1e_1
\]

and define the player’s payoff on first step as

\[
m_1 = \mu_0.
\]

The player’s payoff on second step is

\[
m_2 = \mu_0 + \mu_1.
\]

Further,

\[
m_3 = \mu_0 + \mu_1 + \mu_2, \quad m_5 = \mu_0 + \mu_1 + 4\mu_2,
\]

\[
m_4 = \mu_0 + \mu_1 + 2\mu_2, \quad \ldots
\]

\[
m_5 = \mu_0 + \mu_1 + 3\mu_2, \quad m_n = \mu_0 + \mu_1 + (n-2)\mu_2.
\]

Then normalized player’s payoff for our automaton is

\[
f_{112222} = \lim_{n \to \infty} \frac{m_n}{n} = \mu_2.
\]
4 Analytical solution of 112211-automaton.

This machine has the following transition function

<table>
<thead>
<tr>
<th></th>
<th>$a_0b_0$</th>
<th>$a_0b_1$</th>
<th>$a_1b_0$</th>
<th>$a_1b_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>$S_1$</td>
<td>$S_1$</td>
<td>$S_*$</td>
<td>$S_*$</td>
</tr>
<tr>
<td>$S_1$</td>
<td>$S_2$</td>
<td>$S_2$</td>
<td>$S_*$</td>
<td>$S_*$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$S_1$</td>
<td>$S_1$</td>
<td>$S_*$</td>
<td>$S_*$</td>
</tr>
</tbody>
</table>

and components of the vectors of optimal strategies:

- $p(s_0)$: $c_{0k}^* = (c_{00}, c_{01}) = (c_0, c_1)$,
- $p(s_1)$: $d_{0k}^* = (d_{00}, d_{01}) = (d_0, d_1)$,
- $p(s_2)$: $e_{0k}^* = (e_{00}, e_{01}) = (e_0, e_1)$.

Then

- $m_1 = \mu_0$, $m_5 = \mu_0 + 2\mu_1 + 2\mu_2$,
- $m_2 = \mu_0 + \mu_1$,
- $m_3 = \mu_0 + \mu_1 + \mu_2$,
- $m_4 = \mu_0 + 2\mu_1 + \mu_2$,
- $m_{2n} = \mu_0 + n\mu_1 + (n - 1)\mu_2$,
- $m_{2n+1} = \mu_0 + n\mu_1 + n\mu_2$.

In this case the normalized player’s payoff for our automaton is

$$f_{112211} = \lim_{n \to \infty} \frac{m_n}{n} = \frac{\mu_1 + \mu_2}{2}.$$  

5 Analytical solution of 112200-automaton.

This machine has the following transition function

<table>
<thead>
<tr>
<th></th>
<th>$a_0b_0$</th>
<th>$a_0b_1$</th>
<th>$a_1b_0$</th>
<th>$a_1b_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>$S_1$</td>
<td>$S_1$</td>
<td>$S_*$</td>
<td>$S_*$</td>
</tr>
<tr>
<td>$S_1$</td>
<td>$S_2$</td>
<td>$S_2$</td>
<td>$S_*$</td>
<td>$S_*$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$S_0$</td>
<td>$S_0$</td>
<td>$S_*$</td>
<td>$S_*$</td>
</tr>
</tbody>
</table>

and

- $m_1 = \mu_0$,
- $m_2 = \mu_0 + \mu_1$,
- $m_3 = \mu_0 + \mu_1 + \mu_2$,
- $m_4 = 2\mu_0 + \mu_1 + \mu_2$,
- $m_5 = 2\mu_0 + 2\mu_1 + \mu_2$,
- $m_{3n-2} = n\mu_0 + (n-1)\mu_1 + (n-1)\mu_2$,
- $m_{3n-1} = n\mu_0 + n\mu_1 + (n-1)\mu_2$,
- $m_{3n} = n\mu_0 + n\mu_1 + n\mu_2$.

The normalized player’s payoff this automaton is

$$f_{112200} = \lim_{n \to \infty} \frac{m_n}{n} = \frac{\mu_0 + \mu_1 + \mu_2}{3}.$$
6 Analytical solution of 112212-automaton.

This machine has the following transition function

<table>
<thead>
<tr>
<th></th>
<th>(a_0b_0)</th>
<th>(a_0b_1)</th>
<th>(a_1b_0)</th>
<th>(a_1b_1)</th>
<th>(a_2b_0)</th>
<th>(a_2b_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_0)</td>
<td>(S_1)</td>
<td>(S_1)</td>
<td>(S_*))</td>
<td>(S_*)</td>
<td>(S_0)</td>
<td>(S_0)</td>
</tr>
<tr>
<td>(S_1)</td>
<td>(S_2)</td>
<td>(S_2)</td>
<td>(S_*))</td>
<td>(S_*)</td>
<td>(S_0)</td>
<td>(S_0)</td>
</tr>
<tr>
<td>(S_2)</td>
<td>(S_0)</td>
<td>(S_2)</td>
<td>(S_*))</td>
<td>(S_*)</td>
<td>(S_0)</td>
<td>(S_0)</td>
</tr>
</tbody>
</table>

\[m_3 = \mu_0 + \mu_1 + \mu_2,\]
\[m_4 = \mu_0 + (1 + b_0)\mu_1 + (2 - b_0)\mu_2,\]
\[m_5 = \mu_0 + (1 + 2b_0 - b_0^2)\mu_1,\]
\[+ (3 - 2b_0 + b_0^2)\mu_2,\]
\[m_6 = \mu_0 + (1 + 3b_0 - 2b_0^2 + b_0^3)\mu_1 + (4 - 3b_0 + 2b_0^2 - b_0^3)\mu_2,\]
\[m_7 = \mu_0 + (1 + 4b_0 - 3b_0^2 + 2b_0^3 - b_0^4)\mu_1 + (5 - 4b_0 + 3b_0^2 - 2b_0^3 + b_0^4)\mu_2,\]
\[\ldots\]
\[m_n = \mu_0 + \left(n - 1 - \sum_{k=0}^{n-3} (-)^k (n - k - 2)b_0^k\right)\mu_1\]
\[+ \left(\sum_{k=0}^{n-3} (-)^k (n - k - 2)b_0^k\right)\mu_2.\]

Last equation may be easy calculated with help well-known sum
\[\sum_{k=0}^{n} (-)^k b_0^k = \frac{1 + (-b_0)^n b_0}{1 + b_0}, \quad \sum_{k=0}^{n} (-)^k k b_0^k = b_0 \frac{(-b_0)^n [1 + n + nb_0] - 1}{(1 + b_0)^2}.\]

Then
\[m_n = \mu_0 + \left(\frac{b_0^2(1 + b_0)(n - 1) + b_0 - (-b_0)^n}{b_0 (1 + b_0)^2}\right)\mu_1\]
\[+ \left(\frac{b_0^2(n - 1) + b_0(n - 2) - (-b_0)^n}{b_0 (1 + b_0)^2}\right)\mu_2\]

and the normalized player’s payoff for our automaton can be written as
\[f_{112212} = \lim_{n \to \infty} \frac{m_n}{n} = \frac{1}{1 + b_0} (b_0\mu_1 + \mu_2).\]

7 Analytical solution of 112202-automaton.

This machine has the following transition function

<table>
<thead>
<tr>
<th></th>
<th>(a_0b_0)</th>
<th>(a_0b_1)</th>
<th>(a_1b_0)</th>
<th>(a_1b_1)</th>
<th>(a_2b_0)</th>
<th>(a_2b_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_0)</td>
<td>(S_1)</td>
<td>(S_1)</td>
<td>(S_*)</td>
<td>(S_*)</td>
<td>(S_0)</td>
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</tr>
<tr>
<td>(S_1)</td>
<td>(S_2)</td>
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<td>(S_*)</td>
<td>(S_*)</td>
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<tr>
<td>(S_2)</td>
<td>(S_0)</td>
<td>(S_2)</td>
<td>(S_*)</td>
<td>(S_*)</td>
<td>(S_0)</td>
<td>(S_0)</td>
</tr>
</tbody>
</table>

\[m_4 = (1 + b_0)\mu_0 + \mu_1 + (2 - b_0)\mu_2,\]
\[m_5 = (1 + 2b_0 - b_0^2)\mu_0\]
\[+ (1 + b_0)\mu_1\]
\[+ (3 - 2b_0 + b_0^2)\mu_2,\]
\[ m_6 = (1 + 3b_0 - 3b_0^2 + b_0^3)\mu_0 + (1 + 2b_0 - b_0^2)\mu_1 + (4 - 5b_0 + 4b_0^2 - b_0^3)\mu_2, \]
\[ m_7 = (1 + 4b_0 - 5b_0^2 + 4b_0^3 - b_0^4)\mu_0 + (1 + 3b_0 - 3b_0^2 + b_0^3)\mu_1 \]
\[ + (5 - 7b_0 + 8b_0^2 - 5b_0^3 + b_0^4)\mu_2, \]
and the normalized player’s payoff
\[ f_{112202} = \frac{b_0}{1 + 2b_0}(b_0\mu_0 + b_0\mu_1 + \mu_2). \]

8 Analytical solution of 112201-automaton.

This machine has the following transition function

<table>
<thead>
<tr>
<th>( S_0 )</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S^* )</th>
<th>( S^* )</th>
<th>( S^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0b_0 )</td>
<td>( a_0b_1 )</td>
<td>( a_1b_0 )</td>
<td>( a_1b_1 )</td>
<td>( \ldots )</td>
<td></td>
</tr>
</tbody>
</table>

\[ m_4 = (1 + b_0)\mu_0 + (2 - b_0)\mu_1 + \mu_2, \]
\[ m_5 = (1 + b_0)\mu_0 + 2\mu_1 + (2 - b_0)\mu_2, \]
\[ m_6 = (1 + 2b_0 - b_0^2)\mu_0 \]
\[ + (3 - 2b_0 + b_0^2)\mu_1 + 2\mu_2, \]
and the normalized player’s payoff
\[ f_{112202} = \frac{1}{2 + b_0}(b_0\mu_0 + \mu_1 + \mu_2). \]

9 Conclusions.

In this paper, a complete classification of Bayesian automata type (1) was presented. As a result, we received 57 non-equivalent machines. Notice that if graph of automaton has branched structure then to get an exact solution is not always possible. Therefore, the analysis of specific models should be carried out by methods of situational modeling [3,4]. If the model has an exact solution then its analysis is a simple substitution of the payoff matrix components into analytical expressions.

References


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