A Note on a Characterization of J-Shaped Distribution

by Truncated Moment

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Abstract

Characterization of a probability distribution plays an important role in statistics and mathematical sciences. A probability distribution can be characterized through various methods. The J-shaped distribution is one of the widely used distributions in many fields of research, where the shapes of probability distributions of non-normal data exhibit J-shaped distribution. In this paper, we investigate the characterization of J-shaped distribution by truncated moment. For the sake of completeness, some properties of J-shaped distribution are provided. It is hoped that the findings of this paper will be useful for researchers in different fields of applied sciences.

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1. Introduction

In many fields of research, such as, such as, biology, economics, forestry, genetics, medicine, psychology, reliability, etc., the shapes of probability distributions of non-normal data exhibit J-shaped distributions. Many authors have studied the J-shaped (or Topp–Leone) distribution and its properties. A mathematical formulation of the family of J-shaped probability distributions was first proposed by Topp and Leone [14]. They also derived its first four moments, and showed its suitability to model failure data. Further development continued with the contributions of many authors, among them Nadarajah and Kotz [13], Ghitany, et al. [9], Kotz and Nadarajah [11], Zhou, et al. [17], Zghoul [15, 16], and Genç [8], are notable. Characterization of a probability distribution plays an important role in statistics and mathematical sciences. Before a particular probability distribution model is applied to fit the real world data, it is essential to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. It appears from literature that no attention has been paid on the characterizations of the J-shaped distribution. A probability distribution can be characterized through various methods. In this paper, we have established some characterizations of the J-shaped distribution by using the truncated moment where we have considered a product of reverse hazard rate and another function of the truncated point. The organization of the paper is as follows. For the sake of completeness, some properties of J-shaped distribution are presented in Section 2. In Section 3, we investigate the characterization of the J-shaped distribution by using the truncated moment. Concluding remarks are provided in Section 4.

2. Distributional Properties of the J-Shaped Distribution

For the sake of completeness, some properties of J-shaped distribution are presented in this section. For an absolutely continuous random variable $X$, the family of cumulative distribution functions (cdf ’s) considered by Topp and Leone [14], and exhibiting J-shaped distributions, is defined as follows:

$$F(x) = \begin{cases} 
0, & x < 0, \\
\frac{\alpha x}{\beta} \left(2 - \frac{x}{\beta}\right)^{\gamma} + (1 - \alpha) \frac{x}{\beta}, & 0 \leq x \leq \beta < \infty, \\
1, & x > \beta,
\end{cases}$$

(2.1)

where $0 < \gamma < 1$ and $0 < \alpha \leq 1$. The corresponding family of probability density functions (pdf’s) obtained by differentiation of Eq. (2.1) is given by
Characterization of J-shaped distribution by truncated moment

\[ f(x) = \frac{2\alpha \gamma}{\beta} \left(1 - \frac{x}{\beta}\right) \left(\frac{x}{\beta} \left(2 - \frac{x}{\beta}\right)\right)^{\gamma-1} + \frac{(1 - \alpha)}{\beta}. \]  

For an absolutely continuous random variable \( X \), when \( \alpha = 1 \), we will say that it has the general form of J-shaped distribution, that is, if its distribution function is given by

\[
F(x) = \begin{cases} 
0, & x < 0, \\
\left[\frac{x}{\beta} \left(2 - \frac{x}{\beta}\right)\right]^\gamma, & 0 \leq x \leq \beta < \infty, 0 < \gamma < 1, \\
1, & x > \beta,
\end{cases}
\]

with the probability density function given by

\[
f(x) = \frac{2\gamma}{\beta} \left(1 - \frac{x}{\beta}\right) \left(\frac{x}{\beta} \left(2 - \frac{x}{\beta}\right)\right)^{\gamma-1}. \]  

As pointed out by Topp and Leone [14], the distributions as given above are called J-shaped because \( f(x) > 0 \), \( f'(x) < 0 \), and \( f''(x) > 0 \), for all \( 0 < x < \beta \), where \( f'(x) \) and \( f''(x) \) denote the first and second derivatives of the pdf \( f(x) \) respectively. The shape of the pdf \( f(x) \) as given in Eq. (1.4) is illustrated in the following figure 2.1 for \( \gamma = 0.5, 0.75, 0.95 \), when \( \beta = 1 \).

It can be easily seen that the probability density function (2.4) of J-shaped distribution satisfies the following generalized Pearson system of differential equation

\[
\frac{f''(x)}{f(x)} = \frac{a_0 x + a_1 x^2 + a_2 x^3}{b_0 x + b_1 x^2 + b_2 x^3 + b_3 x^4},
\]

where \( a_0 = 2(\gamma - 1), a_1 = 2(1 - 2\gamma), a_2 = 2\gamma - 1, b_0 = 0, b_1 = 2, b_2 = -3, b_3 = 1. \)
Moments: For any real number $s > 0$, using the general form (2.4), the moment, $E(X^s)$, of order $s$ of the random variable $X$ having the general form of J-shaped distribution is derived as follows.

$$E(X^s) = \frac{2\gamma\beta^s}{\beta} \int_0^\beta \left( \frac{x}{\beta} \right)^{s + \gamma - 1} \left( 1 - \frac{x}{\beta} \right) \left( \frac{x}{\beta} \left( 2 - \frac{x}{\beta} \right) \right)^{\gamma - 1} \, dx. \quad (2.5)$$

Substituting $\frac{x}{\beta} = u$ in Eq. (2.5), and then using the definition of the incomplete beta function, $B_s(a, b)$, which is defined as follows
Characterization of J-shaped distribution by truncated moment

$$B_1(a, b) = \int_0^a \frac{1}{1 - z} \, dz,$$

see, for example, Abramowitz and Stegum [1], and Gradshteyn and Ryzhik [10], we obtain, after simplification, the following expression for the moment:

$$E(X^s) = 2^{s + 2\gamma} \beta^s \left[ B_{1/2} \left( s + \gamma, \gamma \right) - 2 B_{1/2} \left( s + \gamma + 1, \gamma \right) \right].$$  \hspace{1cm} (2.6)

**Integer Order Moments:** For this, considering the Eq. (2.5), and taking \( s = n \), where \( n > 0 \) is an integer, we obtain, after simplification, the following expression for the \( n \)-th order integer moment, \( E(X^n) \):

$$E(X^n) = \beta^n - \beta^n \sum_{j=1}^n \binom{n}{j} (-1)^{n-j} \frac{\Gamma\left(\frac{j+2}{\gamma+1}\right)\Gamma\left(\frac{\gamma+1}{\gamma+1}\right)}{\Gamma\left(\frac{\gamma+1}{\gamma+1}\right)},$$  \hspace{1cm} (2.7)

from which, taking \( n = 1 \) and \( n = 2 \), the first and second moments are respectively given by

$$E(X) = \beta - \beta \frac{\Gamma\left(\frac{2}{\gamma+1}\right)\Gamma\left(\gamma+1\right)}{\Gamma\left(\frac{\gamma+1}{\gamma+1}\right)}, \text{ and } E(X^2) = \beta^2 \frac{\gamma+2}{\gamma+1} - 2\beta^2 \frac{\Gamma\left(\frac{\gamma+1}{\gamma+1}\right)}{\Gamma\left(\frac{2}{\gamma+1}\right)},$$  \hspace{1cm} (2.8)

Consequently, the variance is given by

$$Var(X) = E(X^2) - \left[E(X)\right]^2 = \beta^2 \frac{\gamma+2}{\gamma+1} - \beta^2 \left[\frac{\Gamma\left(\frac{\gamma+1}{\gamma+1}\right)}{\Gamma\left(\frac{2}{\gamma+1}\right)}\right]^2 - \beta^2.$$  \hspace{1cm} (2.9)

It can easily be seen, by direct differentiation, that both moment and variance are increasing functions in \( \gamma \), for fixed \( \beta \).

**Shannon Entropy:** The Shannon entropy of the of J-shaped distribution, when \( \beta = 1 \) in Eq. (2.4), is given by

$$H_X(\gamma) = -(2\gamma) \left( \int_0^1 \left[ (x - x^{-})^{-1} [\ln(2\gamma) + \ln(1-x) + (\gamma-1)\ln(x) + (\gamma-1)\ln(2-x)] - x^{-}(2-x)^{-1} [\ln(2\gamma) + \ln(1-x) + (\gamma-1)\ln(x) + (\gamma-1)\ln(2-x)] \right] dx, \right.$$

from which, on using appropriate substitutions, applying the Equations 4.253.1/P. 538, 5.293.1/P. 557, 8.360/P. 943, and 8.370/P. 947, of Gradsteyn and Ryzhik [10], appropriately, and simplifying, we obtain the following expression for the Shannon entropy:
\[ H_X(\gamma) = -\ln(2\gamma) + \frac{\gamma}{2} B(\gamma, 1)[\psi(\gamma + 1) - \psi(1)] \]
\[ + 2\gamma(\gamma - 1) \sum_{m=0}^{\infty} \binom{\gamma - 1}{m} B(\gamma, m + 2) \left[ \psi(\gamma + m + 2) - \psi(\gamma) \right] \]
\[ + 2\gamma(\gamma - 1) \sum_{m=0}^{\infty} (-1)^m \binom{\gamma - 1}{m} \frac{1}{2m + 2} [\beta(2m + 2) - \ln(2)], \]
(2.10)
where \( \psi(\cdot) \) denotes the psi function, \( B(a, b) \) denotes the beta function, and the function \( \beta(z) \) is defined as follows:
\[ \beta(z) = \frac{1}{2} \left[ \psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right) \right], \]
see, for example, Gradshteyn and Ryzhik [10], among others. It can easily be seen, by direct differentiation, that the Shannon entropy of the J-shaped distribution is a negative, increasing, convex function of \( \gamma \), and, as \( \gamma \to 1 \), we have the entropy \( H_X(\gamma) \to 0 \).

3. A Characterization of the J-Shaped Distribution

A probability distribution can be characterized through various methods (see, for example, Ahsanullah et al. [2], among others). In recent years, there has been a great interest in the characterizations of probability distributions by truncated moments. For example, the development of the general theory of the characterizations of probability distributions by truncated moment began with the work of Galambos and Kotz [4]. Further development on the characterizations of probability distributions by truncated moments continued with the contributions of many authors and researchers, among them Kotz and Shanbhag [12], Glänzel [5, 6], and Glänzel et al. [7], are notable. However, most of these characterizations are based on a simple relationship between two different moments truncated from the left at the same point. As pointed out by Glänzel [5], these characterizations may also serve as a basis for parameter estimation. It appears from literature that no attention has been paid on the characterizations of the J-shaped distribution by using truncated moment. In this section, we present the characterization of the J-shaped distribution by using the truncated moment. We first prove a lemma (Lemma 3.1) which will be useful in proving our main characterization results. The main
characterization results are proved in Theorems 3.1 and 3.2.

**Lemma 3.1:** Suppose that $X$ is an absolutely continuous (with respect to Lebesgue measure) random variable with cdf $F(x)$ and pdf $f(x)$. We assume that $F(0) = 0$, $F(x) > 0 \forall x > 0$, $f'(x)$ exist for all $x \in (0, \infty)$, and $E(X) < \infty$. If $E(X | X \leq t) = g(t) \eta(t)$, where $g(t)$ is a differentiable function of $t$, $0 < t < \infty$, and $\eta(t) = \frac{f(t)}{F(t)}$ for all $t > 0$, then we have

$$f(x) = ce^{\int_{0}^{x} \frac{x-g'(x)}{g(x)} dx},$$

where $c$ is determined by the condition that $\int_{0}^{\infty} f(x)dx = 1$.

**Proof of Lemma 3.1:** We have

$$\int_{0}^{x} uf(u)du \quad \frac{g(x)f(x)}{F(x)}.$$

Thus

$$\int_{0}^{x} uf(u)du = g(x)f(x).$$

Differentiating both sides of the above equation with respect to $x$, we obtain

$$xf(x) = g'(x)f(x) + g(x)f'(x).$$

On simplification, we get

$$f'(x) = \frac{x-g'(x)}{g(x)} f(x).$$

Integrating the above equation, we obtain

$$f(x) = ce^{\int_{0}^{x} \frac{x-g'(x)}{g(x)} dx},$$

where $c$ is determined by the condition that $\int_{0}^{\infty} f(x)dx = 1$.

This completes the proof of Lemma 3.1.

**Theorem 3.1:** Suppose that the random variable $X$ has an absolutely continuous (with respect to Lebesgue measure) cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. We assume that $F(0) = 0$, $F(x) > 0 \forall x > 0$, $f'(x)$ exists for all $x \in (0, \infty)$, and
Let \( E(X) < \infty \). Let \( \eta(t) = \frac{f(t)}{F(t)} \). Then \( X \) has a J-shaped distribution if and only if

\[
E(X \mid X \leq t) = g(t)\eta(t),
\]

where

\[
g(t) = \frac{t^2(2 \beta - t) - \frac{2t}{2\gamma + 1} \left[ B\left(\frac{1}{2}, \gamma + 1\right) - B\left(\frac{1}{2}, \gamma + 1\right) \right]}{4(\beta - t)(2 \beta - t)^{\gamma - 1}}, \quad \beta > 0, \ 0 < \gamma < 1,
\]

and

\[
B_x(p, q) = \int_0^x u^{p-1}(1-u)^{q-1} du \quad \text{(called the incomplete beta function)}.
\]

**Proof of Theorem 3.1:**

**Necessary Part:** Suppose that \( f(x) = \frac{2\gamma}{\beta} \left(1 - \frac{x}{\beta}\right) \left(\frac{x}{\beta} \left(2 - \frac{x}{\beta}\right)\right)^{\gamma - 1} \), \( 0 \leq x \leq 1, \ 0 < \gamma < 1 \). Then, after simplification, we easily have

\[
g(t) = \int_0^t \frac{x^{2\gamma}}{\beta} \left(1 - \frac{x}{\beta}\right) \left(\frac{x}{\beta} \left(2 - \frac{x}{\beta}\right)\right)^{\gamma - 1} dx
\]

\[
= \frac{t^2(2 \beta - t)}{2\gamma(\beta - t)} - \frac{2t}{2\gamma + 1} \left[ B\left(\frac{1}{2}, \gamma + 1\right) - B\left(\frac{1}{2}, \gamma + 1\right) \right]
\]

\[
= \frac{\beta^{2\gamma + 1}}{4(\beta - t)(2 \beta - t)^{\gamma - 1}} \left[ B\left(\frac{1}{2}, \gamma + 1\right) - B\left(\frac{1}{2}, \gamma + 1\right) \right]
\]

where \( B_x(p, q) = \int_0^x u^{p-1}(1-u)^{q-1} du \) denotes the incomplete beta function.

**Sufficiency Part:** We will prove now the “only if” condition the Theorem 3.1. Suppose that

\[
g(t) = \frac{t^2(2 \beta - t)}{2\gamma(\beta - t)} - \frac{2t}{2\gamma + 1} \left[ B\left(\frac{1}{2}, \gamma + 1\right) - B\left(\frac{1}{2}, \gamma + 1\right) \right]
\]

where \( B_x(p, q) = \int_0^x u^{p-1}(1-u)^{q-1} du \) denotes the incomplete beta function.
Then, after simple differentiation and simplification, it is easily seen that
\[ g'(t) = t - g(t) \left( -\frac{1}{\beta - t} + \frac{\gamma - 1}{t} - \frac{\gamma - 1}{2\beta - t} \right). \]

Thus
\[ \frac{t - g'(t)}{g(t)} = -\frac{1}{\beta - t} + \frac{\gamma - 1}{t} - \frac{\gamma - 1}{2\beta - t}. \]

Hence, by Lemma 3.1, we have
\[ \frac{f'(t)}{f(t)} = \frac{t - g'(t)}{g(t)} = -\frac{1}{\beta - t} + \frac{\gamma - 1}{t} - \frac{\gamma - 1}{2\beta - t}, \]
from which, on integration, we have
\[ f(x) = ce^{\int \left( \frac{1}{\beta - t} + \frac{\gamma - 1}{t} - \frac{\gamma - 1}{2\beta - t} \right) dt} = c(\beta - x)(x(2\beta - x))^{\gamma - 1}, \]
where \( c \) is a constant.

Consequently, using the boundary condition \( \int_0^\infty f(x)dx = 1 \) in the above equation, we obtain
\[ f(x) = \frac{2\gamma}{\beta} \left( 1 - \frac{x}{\beta} \right) \left( \frac{x}{\beta} \left( 2 - \frac{x}{\beta} \right) \right)^{\gamma - 1}, \quad 0 \leq x \leq \beta < \infty, \quad 0 < \gamma < 1. \]

This completes the proof of Theorem 3.1.

**Corollary 3.1:** Suppose that the random variable \( X \) has an absolutely continuous (with respect to Lebesgue measure) cumulative distribution function (cdf) \( F(x) \) and probability density function (pdf) \( f(x) \). We assume that \( F(0) = 0, F(x) > 0, \forall x > 0, \quad f'(x) \) exist for all \( x \in (0, \infty) \), and \( E(X) < \infty \). Let \( \eta(t) = \frac{f(t)}{F(t)} \). Then \( X \) has a J-shaped distribution with pdf
\[ f(x) = 2\gamma(1-x)(x(2-x))^{\gamma-1}, \quad 0 \leq x \leq 1, \quad 0 < \gamma < 1, \]
if and only if \( E(X \mid X \leq t) = g(t)\eta(t) \), where
\[ g(t) = \frac{t^2(2-t)}{2\gamma(1-t)} - \frac{\left[ B\left(\frac{1}{2}, \gamma + 1\right) - B\left(\frac{1}{2}, \gamma + 1\right) \right]}{4(1-t)(t(2-t))^{\gamma-1}}. \]
Proof: Taking $\beta = 1$ in Theorem 3.1, the proof of Corollary 3.1 easily follows.

**Corollary 3.2:** Suppose that the random variable $X$ has an absolutely continuous (with respect to Lebesgue measure) cumulative distribution function ($cdf$) $F(x)$ and probability density function ($pdf$) $f(x)$. We assume that $F(0) = 0$, $F(x) > 0$, $\forall x > 0$, $f'(x)$ exist for all $x \in (0, \infty)$, and $E(X) < \infty$. Let $\eta(t) = \frac{f(t)}{F(t)}$. Then $X$ has a J-shaped distribution with pdf

$$f(x) = 2(1-x), 0 \leq x \leq 1,$$

if and only if $E(X | X \leq t) = g(t)\eta(t)$, where

$$g(t) = \frac{t^2(2-t)}{2(1-t)} - \frac{B\left(\frac{1}{2}, 2\right) - B\left(\frac{t-1}{t}, 2\right)}{4(1-t)}.$$

Proof: Taking $\beta = 1$ and $\gamma = 1$ in Theorem 3.1, the proof of Corollary 3.2 easily follows.

**4. Concluding Remarks**

Before a particular probability distribution model is applied to fit the real world data, it is essential to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. Thus, characterization of a probability distribution plays an important role in statistics and mathematical sciences. A probability distribution can be characterized through various methods. In this paper, we investigate the characterization of J-shaped distribution by truncated moment where we have considered a product of reverse hazard rate and another function of the truncated point. For the sake of completeness, some properties of J-shaped distribution are provided. We believe that the findings of this paper would be useful for the practitioners in various fields of studies, and further enhancement of research in distribution theory and its applications.

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