Identities of Symmetry for $q$-Euler Polynomials
Derived from Fermionic Integral on $\mathbb{Z}_p$
under Symmetry Group $S_3$

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Abstract

The purpose of this paper is to give basic identities of symmetry in three variables related to \(q\)-Euler polynomials and alternating \(q\)-power sums arising from fermionic integral on \(\mathbb{Z}_p\).

1. Introduction

Let \(p\) be a fixed prime with \(p \equiv 1(\text{mod } 2)\). Throughout this paper, \(\mathbb{Z}_p\), \(\mathbb{Q}_p\) and \(\mathbb{C}_p\) will, respectively, denote the ring of \(p\)-adic integers, the field of \(p\)-adic rational numbers and the completion of algebraic closure of \(\mathbb{Q}_p\). Let \(\nu_p\) be the normalized exponential valuation of \(\mathbb{C}_p\) with \(|p|_p = p^{-\nu_p(p)} = 1/p\). Let \(q\) be an indeterminate in \(\mathbb{C}_p\) with \(|1-q|_p < 1\) and the \(q\)-extension of \(x\) is defined as \([x]_q = (1-q^x)/(1-q)\). Note that \(\lim_{q \to 1}[x]_q = x\). Let \(C(\mathbb{Z}_p)\) be the space of continuous functions on \(\mathbb{Z}_p\). For \(f \in C(\mathbb{Z}_p)\), the fermionic integral on \(\mathbb{Z}_p\) is defined by T. Kim to be

\[
I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}} f(x)(-1)^x \quad (\text{see } [2, 3]) \quad (1)
\]

From (1), we note that

\[
I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \quad (2)
\]

where \(f_1(x) = f(x+1)\) (see [2, 3]). By (2), we easily get

\[
I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2 \sum_{x=0}^{n-1} (-1)^{n-1+x} f(x), \quad (3)
\]

where \(f_n(x) = f(x+n)\) (see [2, 3, 4]).

As is well known, the Euler polynomials are defined by the generating function to be

\[
\left(\frac{e^t + 1}{e^t + 1}\right) e^{xt} = e^{\mathcal{E}(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (4)
\]

with the usual convention about replacing \(E^n(x)\) by \(E_n(x)\) (see [1-10]).

When \(x = 0\), \(E_n = E_n(0)\) is called the \(n\)-th Euler number. From (4), we can derive the following recurrence relation:

\[
E_0 = 1, (E + 1)^n + E_n = 2\delta_{0,n} \quad (n \geq 0) \quad (5)
\]

with the usual convention about replacing \(E^n\) by \(E_n\). By (4), we easily see that

\[
E_n(x) = \sum_{m=0}^{n} \binom{n}{m} E_m x^{n-m} \quad (\text{see } [1-10]).
\]
Let us take \( f(x) = e^{xt} \). Then, by (2), we get
\[
\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.
\]
and
\[
\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\]
In the viewpoints of (6) and (7), we consider the \( q \)-extension of Euler numbers and polynomials as follows:
\[
\int_{\mathbb{Z}_p} e^{[x]q^t} d\mu_{-1}(x) = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}.
\]
and
\[
\int_{\mathbb{Z}_p} e^{[x+y]q^t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.
\]
Thus, from (8) and (9), we have
\[
2 \sum_{m=0}^{\infty} (-1)^m e^{[m]q^t} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}.
\]
and
\[
2 \sum_{m=0}^{\infty} (-1)^m e^{[m+x]q^t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.
\]
By (10) and (11), we get
\[
E_{n,q}(x) = \frac{2}{(1-q)^n} \sum_{m=0}^{n} \binom{n}{m} (-1)^m q^{mx} \frac{1}{1+q^m}
\]
and
\[
E_{n,q}(x) = \sum_{m=0}^{n} \binom{n}{m} q^{mx} E_{m,q}[x]_q^{n-m}.
\]
From (2), (8) and (13), we have
\[
E_{0,q} = 1, \ (qE_q + 1)^n + E_{n,q} = 2\delta_{0,n} \ (n \geq 0)
\]
with the usual convention about replacing \( E_q^n \) by \( E_{n,q} \). Thus, we note that
\[
\lim_{q \to 1} E_{n,q} = E_n, \ \text{and} \ \lim_{q \to 1} E_{n,q}(x) = E_n(x).
\]
Recently, D. S. Kim et al gave the symmetry identities of the ordinary Euler and Bernoulli polynomials in three variables (see [7, 9, 11]).

In this paper, we give basic identities of symmetry in three variables related to \( q \)-Euler polynomials and \( q \)-analogue of alternating sums of powers of consecutive \( q \)-integers which are derived from fermionic integral on \( \mathbb{Z}_p \) under symmetry group \( S_3 \).
2. Symmetry identities for $q$-Euler polynomials under symmetry group $S_3$

Let $w_1, w_2, w_3$ be natural numbers with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, $w_3 \equiv 1 \pmod{2}$. Then by (1) and (9), we get

$$I = \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} \int_{Z_p} e^{[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j] t} d\mu_1(y).$$

From (14) and (15), we have

$$I = \lim_{N \to \infty} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} \sum_{k=0}^{w_1-1} \sum_{y=0}^{P^N-1} e^{[w_2 w_3 (k+w_1 y) + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j] t} (-1)^{y+k}.\tag{16}$$

By the same method as (16), we get

$$I = \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} \int_{Z_p} e^{[w_1 w_2 y + w_1 w_2 w_3 x + w_2 w_1 i + w_2 w_3 j] t} d\mu_1(y),\tag{17}$$

$$I = \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} \int_{Z_p} e^{[w_1 w_2 y + w_1 w_2 w_3 x + w_3 w_2 i + w_3 w_1 j] t} d\mu_1(y),\tag{18}$$

$$I = \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} \int_{Z_p} e^{[w_1 w_2 y + w_1 w_2 w_3 x + w_3 w_2 i + w_3 w_1 j] t} d\mu_1(y),\tag{19}$$

$$I = \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} \int_{Z_p} e^{[w_1 w_2 y + w_1 w_2 w_3 x + w_3 w_2 i + w_3 w_1 j] t} d\mu_1(y),\tag{20}$$

and

$$I = \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} (-1)^{i+j} \int_{Z_p} e^{[w_1 w_3 y + w_1 w_2 w_3 x + w_2 w_3 i + w_1 w_2 j] t} d\mu_1(y).\tag{21}$$

Therefore, by (15), (17), (18), (19), (20) and (21), we obtain the following theorem.
Theorem 2.1. For \( w_1, w_2, w_3 \in \mathbb{N} \) with \( w_1 \equiv 1 \pmod{2} \), \( w_2 \equiv 1 \pmod{2} \), \( w_3 \equiv 1 \pmod{2} \), we have

\[
\begin{align*}
    & \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} \int_{\mathbb{Z}_p} e^{[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]q^t} d\mu_{-1}(y) \\
    &= \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} (-1)^{i+j} \int_{\mathbb{Z}_p} e^{[w_3 w_1 y + w_1 w_2 w_3 x + w_2 w_1 i + w_2 w_3 j]q^t} d\mu_{-1}(y) \\
    &= \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} \int_{\mathbb{Z}_p} e^{[w_1 w_2 y + w_1 w_2 w_3 x + w_3 w_2 i + w_3 w_1 j]q^t} d\mu_{-1}(y).
\end{align*}
\]

Now, we observe that

\[
[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q = [w_2 w_3]_q \left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_{q^{w_2 w_3}}.
\]

(22)

Therefore, by (22), we obtain the following theorem.

Theorem 2.2. For \( n \geq 0 \), We have

\[
\begin{align*}
    & [w_2 w_3]_q^{w_1-1} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} (-1)^{i+j} \int_{\mathbb{Z}_p} \left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]^n d\mu_{-1}(y) \\
    &= [w_3 w_1]_q^{w_2-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} \int_{\mathbb{Z}_p} \left[ y + w_2 x + \frac{w_2}{w_3} i + \frac{w_2}{w_1} j \right]^n d\mu_{-1}(y) \\
    &= [w_1 w_2]_q^{w_3-1} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} \int_{\mathbb{Z}_p} \left[ y + w_3 x + \frac{w_3}{w_1} i + \frac{w_3}{w_2} j \right]^n d\mu_{-1}(y),
\end{align*}
\]

where \( w_1, w_2, w_3 \in \mathbb{N} \) with \( w_1 \equiv 1 \pmod{2} \), \( w_2 \equiv 1 \pmod{2} \), \( w_3 \equiv 1 \pmod{2} \).

From (9) and Theorem 2.2, we can derive the following theorem.
Theorem 2.3. For $n \geq 0, w_1, w_2, w_3 \in \mathbb{N}$ with $w_1 \equiv 1(\text{mod } 2)$, $w_2 \equiv 1(\text{mod } 2)$, $w_3 \equiv 1(\text{mod } 2)$, we have

$$\left[w_2 w_3\right]_q^n \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} E_{n,q^{w_2 w_3}} \left(w_1 x + \frac{w_1 i + w_1 j}{w_2 w_3}\right)$$

$$= [w_3 w_1]_q^n \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} (-1)^{i+j} E_{n,q^{w_3 w_1}} \left(w_2 x + \frac{w_2 i + w_2 j}{w_3 w_1}\right)$$

$$= [w_1 w_2]_q^n \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} E_{n,q^{w_1 w_2}} \left(w_3 x + \frac{w_3 i + w_3 j}{w_1 w_2}\right).$$

By (13), we get

$$E_{n,q^{w_2 w_3}} \left(w_1 x + \frac{w_1 i + w_1 j}{w_2 w_3}\right) = \sum_{l=0}^{n} \left(\frac{n}{l}\right) q^{(n-l) w_2 w_3 x} E_{n-l,q^{w_2 w_3}} \left[w_1 x + \frac{w_1 i + w_1 j}{w_2 w_3}\right]^l. \tag{23}$$

It is easy to show that

$$\left[w_1 x + \frac{w_1 i + w_1 j}{w_2 w_3}\right]_{q^{w_2 w_3}} = [w_1 x]_{q^{w_2 w_3}} + q^{w_1 w_2 w_3 x} \left[w_1 i\right]_{q^{w_2 w_3}} + q^{w_1 w_2 w_3 x + w_1 w_3 i} \left[w_1 j\right]_{q^{w_2 w_3}}. \tag{24}$$

From (23) and (24), we have

$$E_{n,q^{w_2 w_3}} \left(w_1 x + \frac{w_1 i + w_1 j}{w_2 w_3}\right) = \sum_{l=0}^{n} \left(\frac{n}{l}\right) q^{(n-l) w_2 w_3 x} E_{n-l,q^{w_2 w_3}}$$

$$\times \sum_{l_1+l_2+l_3=l} \left(l, l_1, l_2, l_3\right) q^{w_1 w_2 w_3 x l_2 + (w_1 w_2 w_3 x + w_1 w_3 i) l_3}$$

$$\times \left[w_1 x\right]_{q^{w_2 w_3}}^{l_1} \left[w_1 i\right]_{q^{w_2 w_3}}^{l_2} \left[w_1 j\right]_{q^{w_2 w_3}}^{l_3}. \tag{25}$$

Therefore, by Theorem 2.3 and (25), we obtain the following theorem.
Theorem 2.4. For \( n \geq 0, w_1, w_2, w_3 \in \mathbb{N} \) with \( w_1 \equiv 1 \pmod{2} \), \( w_2 \equiv 1 \pmod{2} \), \( w_3 \equiv 1 \pmod{2} \), we have

\[
[w_2 w_3]^n q \sum_{l=0}^{n} \sum_{l_1+l_2+l_3=l} \sum_{j=0}^{w_2-1} \sum_{j_1=0}^{w_3-1} \binom{n}{l} \binom{l}{l_1, l_2, l_3} q^{(n-l)w_2 w_3} E_{n-l, q^{w_2 w_3}} (-1)^i j
\times q^{w_1 w_2 w_3 x l_2 + (w_1 w_2 w_3 x + w_1 w_3 i) l_1} [w_1 x]_{q^{w_2 w_3}}^{l_1} [w_1 w_2] [w_1 w_3]^{l_2} [w_1 w_2]^{l_3} q^{w_2 w_3}
= [w_2 w_1]_q^n \sum_{l=0}^{n} \sum_{l_1+l_2+l_3=l} \sum_{j=0}^{w_3-1} \sum_{j_1=0}^{w_1-1} \binom{n}{l} \binom{l}{l_1, l_2, l_3} q^{(n-l)w_3 w_1} E_{n-l, q^{w_1 w_3}} (-1)^i j
\times q^{w_1 w_2 w_3 x l_2 + (w_1 w_2 w_3 x + w_1 w_1 i) l_1} [w_1 w_2 x]_{q^{w_1 w_3}}^{l_1} [w_2 w_1]^{l_2} [w_1 w_2]^{l_3} q^{w_2 w_1}
= [w_2 w_1]_q^n \sum_{l=0}^{n} \sum_{l_1+l_2+l_3=l} \sum_{j=0}^{w_2-1} \sum_{j_1=0}^{w_1-1} \binom{n}{l} \binom{l}{l_1, l_2, l_3} q^{(n-l)w_1 w_2} E_{n-l, q^{w_1 w_2}} (-1)^i j
\times q^{w_1 w_2 w_3 x l_2 + (w_1 w_2 w_3 x + w_1 w_2 i) l_1} [w_3 x]_{q^{w_1 w_2}}^{l_1} [w_3 w_1]^{l_2} [w_3 w_2]^{l_3} q^{w_1 w_2}.
\]

We observe that

\[
[y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j]_{q^{w_2 w_3}} = \frac{[w_1]_q}{[w_2 w_3]_q} [w_3 i + w_2 j]_{q^{w_1}} + q^{w_1 w_3 i + w_2 w_3 j} [y + w_1 x]_{q^{w_2 w_3}}.
\]

(26)

From (26), we have

\[
\int_{\mathbb{Z}_p} \left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_q^n d\mu_{-1}(y)
= \sum_{k=0}^{n} \binom{n}{k} \left( \frac{[w_1]_q}{[w_2 w_3]_q} \right)^{n-k} [w_3 i + w_2 j]_{q^{w_1}}^{n-k} q^{k(w_1 w_3 i + w_1 w_2 j)} \int_{\mathbb{Z}_p} [y + w_1 x]_{q^{w_2 w_3}}^k d\mu_{-1}(y)
= \sum_{k=0}^{n} \binom{n}{k} \left( \frac{[w_1]_q}{[w_2 w_3]_q} \right)^{n-k} [w_3 i + w_2 j]_{q^{w_1}}^{n-k} q^{k(w_1 w_3 i + w_1 w_2 j)} E_{k, q^{w_2 w_3}} (w_1 x).
\]

(27)
Thus, by (27), we get
\[ [w_2 w_3]^n_q \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} \int_{Z_p} \left[ y + w_1 x + \frac{w_1}{w_2} + \frac{w_1}{w_3} \right]^{n_q w_{23}} d\mu_{-1}(y) \]
\[ = \sum_{k=0}^{n} \binom{n}{k} [w_2 w_3]^k_q [w_1]^{n-k}_q E_{k,q^{w_{23}}} (w_1 x) \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} q^{(w_1 w_2 i + w_1 w_2 j)k} \]
\[ \times [w_3^i + w_3^j]^{n-k}_q \]
\[ = \sum_{k=0}^{n} \binom{n}{k} [w_2 w_3]^k_q [w_1]^{n-k}_q E_{k,q^{w_{23}}} (w_1 x) \tilde{T}_{n,q^{w_1}} (w_2, w_3 | k), \]
where
\[ T_{n,q}(w_1, w_2 | k) = \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} q^{(w_2 i + w_1 j)k} (-1)^{i+j} [w_2 i + w_1 j]^{n-k}_q. \]

By the same method as (28), we get
\[ [w_3 w_1]^n_q \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} \int_{Z_p} \left[ y + w_2 x + \frac{w_2}{w_3} + \frac{w_2}{w_1} \right]^{n_q w_{12}} d\mu_{-1}(y) \]
\[ = \sum_{k=0}^{n} \binom{n}{k} [w_1 w_3]^k_q [w_2]^{n-k}_q E_{k,q^{w_{12}}} (w_2 x) \tilde{T}_{n,q^{w_2}} (w_1, w_2 | k) \]
and
\[ [w_1 w_2]^n_q \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} \int_{Z_p} \left[ y + w_3 x + \frac{w_3}{w_1} + \frac{w_3}{w_2} \right]^{n_q w_{12}} d\mu_{-1}(y) \]
\[ = \sum_{k=0}^{n} \binom{n}{k} [w_1 w_3]^k_q [w_3]^{n-k}_q E_{k,q^{w_{12}}} (w_3 x) \tilde{T}_{n,q^{w_2}} (w_1, w_2 | k). \]

Therefore, by Theorem 2.2, (28), (30) and (31), we obtain the following theorem.

**Theorem 2.5.** For \( n \geq 0 \), \( w_1, w_2, w_3 \in \mathbb{N} \) with \( w_1 \equiv 1 (mod 2) \), \( w_2 \equiv 1 (mod 2) \), \( w_3 \equiv 1 (mod 2) \), we have
\[ \sum_{k=0}^{n} \binom{n}{k} [w_2 w_3]^k_q [w_1]^{n-k}_q E_{k,q^{w_{23}}} (w_1 x) \tilde{T}_{n,q^{w_1}} (w_2, w_3 | k) \]
\[ = \sum_{k=0}^{n} \binom{n}{k} [w_1 w_3]^k_q [w_2]^{n-k}_q E_{k,q^{w_{12}}} (w_2 x) \tilde{T}_{n,q^{w_2}} (w_3, w_1 | k) \]
\[ = \sum_{k=0}^{n} \binom{n}{k} [w_1 w_2]^k_q [w_3]^{n-k}_q E_{k,q^{w_{12}}} (w_3 x) \tilde{T}_{n,q^{w_2}} (w_1, w_2 | k). \]
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