Identities of Symmetry for Generalized Higher-Order $q$-Euler Polynomials under $S_3$

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Abstract

In this paper, we study the identities of symmetry for the generalized higher-order $q$-Euler polynomials in three variable under symmetry group $S_3$ which are derived from the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$. 
1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $| \cdot |_p$ be the normalized $p$-adic absolute value with $|p|_p = 1/p$ and let $q$ be an indeterminate in $\mathbb{C}_p$ such that $|1 - q|_p < p^{-1/(p-1)}$. The $q$-analogue of $x$ is defined by $[x]_q = (1 - q^x)/(1 - q)$. Note that $\lim_{q \to 1} [x]_q = x$. Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then the fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_q(-q(f)) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)(-q)^x,$$

where $[x]_{-q} = (1 - (-q)^x)/(1 + q)$, (see [6, 7, 10, 11]). For $n \geq 1$, by (1), we get

$$q^n \int_{\mathbb{Z}_p} f(x + n) d\mu_{-q}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2]_q \sum_{\ell=0}^{n-1} q^\ell f(\ell)(-1)^{n-1-\ell}. \tag{2}$$

In particular, for $n = 1$, we have

$$q \int_{\mathbb{Z}_p} f(x + 1) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2]_q f(0), \quad \text{(see [6, 7]).} \tag{3}$$

The $q$-Euler polynomials are defined by Kim to be

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [y + x]^n d\mu_{-q}(y), \quad (n \geq 0), \quad \text{(see [1 - 18]).} \tag{4}$$

When $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called the $q$-Euler numbers. For $d \in \mathbb{N}$ with $(d, p) = 1$ and $d \equiv 1 (\text{mod } 2)$, we set

$$\lim_{N} \mathbb{Z}/dp^N\mathbb{Z}, \quad X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} (a + dp\mathbb{Z}_p)$$

and

$$a + dp^N\mathbb{Z}_p = \{ x \in X \mid x \equiv a (\text{mod } dp^N) \},$$

where $a \in \mathbb{Z}$ lies $0 \leq a < dp^N$.

Let $\chi$ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod } 2)$. Then the generalized $q$-Euler polynomials attached to $\chi$ are defined by Kim to be

$$\int_X \chi(y)[x + y]^n d\mu_{-q}(y) = E_{n,q,\chi}[x], \quad (n \geq 0).$$

When $x = 0$, $E_{n,q,\chi} = E_{n,q,\chi}(0)$ are called the generalized $q$-Euler numbers attached to $\chi$, (see [6, 7]).
In this paper, we consider the generalized $q$-Euler polynomials attached to $\chi$ and study some symmetric identities for the generalized higher-order $q$-Euler polynomials in three variables under symmetry group $S_3$ which are derived from the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$.

2. SYMMETRY IDENTITIES FOR THE GENERALIZED HIGHER-ORDER $q$-EULER POLYNOMIALS

For $r \in \mathbb{N}$, let us consider the generalized higher-order $q$-Euler polynomials attached to $\chi$ as follows:

$$\int_X \cdots \int_X \prod_{\ell=1}^r (\chi(x_\ell)) [x_1 + \cdots + x_r + x]_q^n d\mu_q(x_1) \cdots d\mu_q(x_r) = \sum_{n=0}^{\infty} \frac{E_n^{(r)}(x)}{n!} \chi(x), \quad \text{(see [4, 6, 7])},$$

(5)

Thus, by (5), we get

$$\int_X \cdots \int_X \prod_{\ell=1}^r (\chi(x_\ell))[x_1 + \cdots + x_r + x]_q^n d\mu_q(x_1) \cdots d\mu_q(x_r) = E_n^{(r)}(x), \quad (n \geq 0).$$

(6)

When $x = 0$, $E_n^{(r)}(0)$ are called the generalized higher-order $q$-Euler numbers attached to $\chi$.

Let $w_1, w_2, w_3 \in \mathbb{N}$ with $w_i \equiv 1(\bmod 2), \ (i = 1, 2, 3)$. Then, by (1) and (5), we get

$$\int_X \cdots \int_X \prod_{\ell=1}^r (\chi(x_\ell))
\times e^{[w_2 w_3 \sum_{\ell=1}^{r} x_\ell + w_1 w_2 w_3 x + w_1 w_3 \sum_{\ell=1}^{r} i_{\ell} + w_1 w_2 \sum_{\ell=1}^{r} j_{\ell}]q^t} d\mu_{q^{-w_2 w_3}}(x_1) \cdots d\mu_{q^{-w_2 w_3}}(x_r)
= \lim_{N \to \infty} \left( \frac{1}{[dp^N]_{-q^{w_2 w_3}}} \right)^r \sum_{x_1, \cdots, x_r = 0} \prod_{\ell=1}^r \chi(x_\ell)
\times e^{[w_2 w_3 \sum_{\ell=1}^{r} x_\ell + w_1 w_2 w_3 x + w_1 w_3 \sum_{\ell=1}^{r} i_{\ell} + w_1 w_2 \sum_{\ell=1}^{r} j_{\ell}]q^t} (-q^{w_2 w_3})^{x_1 + \cdots + x_r}
= \lim_{N \to \infty} \left( \frac{1}{[w_1 dp^N]_{-q^{w_2 w_3}}} \right)^r \sum_{k_1, \cdots, k_r = 0} \sum_{y_1, \cdots, y_r = 0} \prod_{\ell=1}^r \chi(k_\ell)
\times q^{w_2 w_3 \sum_{\ell=1}^{r} (k_\ell + w_1 y_\ell)} e^{[w_2 w_3 \sum_{\ell=1}^{r} (k_\ell + dw_1 y_\ell) + w_1 w_2 w_3 x + w_1 w_3 \sum_{\ell=1}^{r} i_{\ell} + w_1 w_2 \sum_{\ell=1}^{r} j_{\ell}]q^t},$$

(7)

where $d \in \mathbb{N}$ with $d \equiv 1(\bmod 2)$. By (7), we get
Therefore, by Theorem 1 and (9), we obtain the following theorem.

\[(\frac{1}{[w_2w_3]_q})^r \sum_{i_1,\ldots,i_r=0}^{d-w_1} \sum_{j_1,\ldots,j_r=0}^{d-w_2w_3} \left( \prod_{\ell=1}^{r} (\chi(i_\ell j_\ell)) \right) (-1)^{\sum_{\ell=1}^{r}(i_\ell+j_\ell)} \times q^{w_2w_3} \sum_{\ell=1}^{r} i_\ell + w_1 w_2 \sum_{\ell=1}^{r} j_\ell \int_X \ldots \int_X \left( \prod_{\ell=1}^{r} (\chi(x_\ell)) \right) \times e^{\left[ w_2w_3 \sum_{\ell=1}^{r} x_\ell + w_1 w_2w_3 x + w_1w_3 \sum_{\ell=1}^{r} i_\ell + w_1 w_2 \sum_{\ell=1}^{r} j_\ell \right]_q t} \times d\mu_{-q}^{w_2w_3} (x_1) \cdots d\mu_{-q}^{w_2w_3} (x_r) \]

As this expression is invariant under any permutation \(\sigma \in S_3\), we have the following theorem.

**Theorem 2.1.** For \(w_1, w_2, w_3, d \in \mathbb{N}\) with \(w_i \equiv 1 (\text{mod } 2), d \equiv 1 (\text{mod } 2), (i = 1, 2, 3)\), the following expressions

\[(\frac{1}{[w_{\sigma(2)}w_{\sigma(3)}]_q})^r \sum_{i_1,\ldots,i_r=0}^{d-w_{\sigma(1)}} \sum_{j_1,\ldots,j_r=0}^{d-w_{\sigma(2)}} \left( \prod_{\ell=1}^{r} (\chi(i_\ell j_\ell)) \right) (-1)^{\sum_{\ell=1}^{r}(i_\ell+j_\ell)} \times q^{w_{\sigma(1)}w_{\sigma(2)}} \sum_{\ell=1}^{r} i_\ell + w_{\sigma(1)} w_{\sigma(2)} \sum_{\ell=1}^{r} j_\ell \int_X \ldots \int_X \left( \prod_{\ell=1}^{r} (\chi(x_\ell)) \right) \times e^{\left[ w_{\sigma(1)} w_{\sigma(2)} (x_\ell + w_{\sigma(1)} w_{\sigma(2)} x) + w_{\sigma(1)} w_{\sigma(2)} \sum_{\ell=1}^{r} i_\ell + w_{\sigma(1)} w_{\sigma(2)} \sum_{\ell=1}^{r} j_\ell \right]_q t} \times d\mu_{-q}^{w_{\sigma(1)} w_{\sigma(2)}} (x_1) \cdots d\mu_{-q}^{w_{\sigma(1)} w_{\sigma(2)}} (x_r) \]

are the same for any \(\sigma \in S_3\).

By (6), we get

\[\int_X \ldots \int_X \left( \prod_{\ell=1}^{r} (\chi(x_\ell)) \right) e^{\left[ w_2w_3 \sum_{\ell=1}^{r} x_\ell + w_1 w_2w_3 x + w_1w_3 \sum_{\ell=1}^{r} i_\ell + w_1 w_2 \sum_{\ell=1}^{r} j_\ell \right]_q t} \times d\mu_{-q}^{w_2w_3} (x_1) \cdots d\mu_{-q}^{w_2w_3} (x_r) \]

\[= \sum_{n=0}^{\infty} \left[ w_2w_3 \right]_q^n \frac{t^n}{n!} \left( w_1 x + \frac{w_1}{w_2} \sum_{\ell=1}^{r} i_\ell + \frac{w_1}{w_3} \sum_{\ell=1}^{r} j_\ell \right) \]

Therefore, by Theorem 1 and (9), we obtain the following theorem.
Theorem 2.2. Let \( w_1, w_2, w_3, d \in \mathbb{N} \) with \( w_1 \equiv 1 \pmod{2} \), \( w_2 \equiv 1 \pmod{2} \), \( w_3 \equiv 1 \pmod{2} \), \( d \equiv 1 \pmod{2} \) and \( n \in \mathbb{N} \cup \{0\} \). Then the following expressions

\[
\left[ \frac{w_{\sigma(2)} w_{\sigma(3)}}{w_{\sigma(2)} w_{\sigma(3)}} \right]^n_q \sum_{\ell=1}^r \sum_{i=0}^{w_{\sigma(2)}-1} \sum_{j=0}^{w_{\sigma(3)}-1} (-1)^{\sum_{\ell=1}^r (i+2\ell)} \prod_{\ell=1}^r \chi(i\ell) \chi(j\ell)
\]

\[
\times q^{w_{\sigma(1)} w_{\sigma(3)} \sum_{\ell=1}^r i_\ell + w_{\sigma(1)} w_{\sigma(2)} \sum_{\ell=1}^r j_\ell} E_{n, q, w_{\sigma(2)}, w_{\sigma(3)}, \chi}^{(r)} \left( \frac{w_{\sigma(1)}}{w_{\sigma(2)}} \sum_{\ell=1}^r i_\ell + \frac{w_{\sigma(1)}}{w_{\sigma(3)}} \sum_{\ell=1}^r j_\ell \right)
\]

are the same for any \( \sigma \in S_3 \).

From (5), we have

\[
\int x \cdots \int x \left( \prod_{\ell=1}^r \chi(x_{\ell}) \right) \left[ \sum_{\ell=1}^r \sum_{\ell=1}^r i_\ell + w_1 \sum_{\ell=1}^r j_\ell \right]^n q^{w_{\sigma(1)} w_{\sigma(3)} \sum_{\ell=1}^r i_\ell + w_{\sigma(1)} w_{\sigma(2)} \sum_{\ell=1}^r j_\ell} \chi(w_1 x) \chi(w_2 x) \chi(w_3 x)
\]

\[
= \sum_{k=0}^n \binom{n}{k} \left( \frac{[w_1]^q}{[w_2 w_3]^q} \right)^{n-k} \left[ \sum_{\ell=1}^r i_\ell + \sum_{\ell=1}^r j_\ell \right]^{n-k} \int x \cdots \int x \left( \sum_{\ell=1}^r \sum_{\ell=1}^r i_\ell + w_1 \sum_{\ell=1}^r j_\ell \right) q^{w_{\sigma(1)} w_{\sigma(3)} \sum_{\ell=1}^r i_\ell + w_{\sigma(1)} w_{\sigma(2)} \sum_{\ell=1}^r j_\ell} \chi(w_1 x) \chi(w_2 x) \chi(w_3 x)
\]

By (10), we get

\[
\left[ \frac{w_2 w_3^n}{w_2 w_3} \right] q^{d w_2 - 1} \sum_{i_1, \cdots, i_r=0}^{w_2 w_3} (-1)^{\sum_{\ell=1}^r (i+2\ell)} \prod_{\ell=1}^r \chi(i\ell) \chi(j\ell)
\]

\[
\times q^{w_{\sigma(1)} w_{\sigma(3)} \sum_{\ell=1}^r i_\ell + w_{\sigma(1)} w_{\sigma(2)} \sum_{\ell=1}^r j_\ell} \int x \cdots \int x \left( \sum_{\ell=1}^r \sum_{\ell=1}^r i_\ell + \sum_{\ell=1}^r j_\ell \right) q^{w_{\sigma(1)} w_{\sigma(3)} \sum_{\ell=1}^r i_\ell + w_{\sigma(1)} w_{\sigma(2)} \sum_{\ell=1}^r j_\ell} \chi(w_1 x) \chi(w_2 x) \chi(w_3 x)
\]

\[
= \sum_{k=0}^n \binom{n}{k} \left( \frac{[w_2 w_3]^q}{[w_2 w_3]^q} \right)^{n-k} E_{n, q, w_{\sigma(2)}, \chi}^{(r)} (w_1 x) I_{n, k, q, \chi}^{(r)} (w_2, w_3 : d|x),
\]
where

\[ T_{n,k,q}^{(r)}(w_1, w_2 : d|\chi) = \sum_{i_1, \ldots, i_r=0}^{d w_1-1} \sum_{j_1, \ldots, j_r=0}^{d w_2-1} q^{w_2 \sum_{\ell=1}^r i_{\ell} + w_1 \sum_{\ell=1}^r j_{\ell}} (k+1) \left( -1 \right)^{\sum_{\ell=1}^r (i_{\ell}+j_{\ell})} \]

\[ \times \left( \prod_{\ell=1}^{r} \chi(i_{\ell}) \chi(j_{\ell}) \right) \left[ w_2 \sum_{\ell=1}^{r} i_{\ell} + w_1 \sum_{\ell=1}^{r} j_{\ell} \right]_{q}^{n-k}. \]  

(12)

Therefore, by (11) and (12), we obtain the following theorem.

**Theorem 2.3.** For \( w_1, w_2, w_3, d \in \mathbb{N} \) with \( w_i \equiv 1 (\text{mod} \ 2), d \equiv 1 (\text{mod} \ 2), (i = 1, 2, 3) \) and \( n \in \mathbb{N} \cup \{0\} \), the following expression

\[ \sum_{k=0}^{n} \binom{n}{k} \left[ w_{\sigma(2)} w_{\sigma(3)} \right]_{q}^{k} \left[ w_{\sigma(1)} \right]_{q}^{n-k} E_{k,q,w_{\sigma(2)},w_{\sigma(3)},\chi}^{(r)} (w_{\sigma(1)} x) T_{n,k,q,w_{\sigma(1)} w_{\sigma(2)},w_{\sigma(3)} : d|\chi}^{(r)} \]  

are the same for any permutation \( \sigma \in S_3 \).

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