Horizontal Weak Compatibility of Mappings and Common Fixed Points in Dislocated Fuzzy Metric Space

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Abstract

The concept of Horizontally Weak Compatibility of mappings is introduced and common fixed point theorems for two pairs of mappings are proved in dislocated fuzzy metric space using horizontally weak compatibility condition. Our results extends and generalises many existing fixed point theorems.
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1 Introduction

In 1965, Zadeh[27] introduced the notion of Fuzzy sets. Since then many fixed point theorems for contractions in fuzzy metric spaces and Quasi fuzzy metric spaces appeared (see[2],[4-9],[11],[14],[15],[16],[17],[21],[22],[24-26]). Hitzler and Seda [10] introduced the concept of dislocated metric space and studied the dislocated topologies associated with it, which is a generalisation of the conventional topologies and can be thought of as underlying the notion of dislocated metrics. They also proved a generalized version of Banach contraction mapping theorem which was applied to obtain fixed point semantics for logic programs. Later Reny George and M.S Khan [17] introduced the concept of dislocated fuzzy metric spaces and studied the associated topologies. Further results on fixed points in dislocated fuzzy metric space were proved in [18],[19] and [20]. Recently, Cho et al[2], Abbas et al[1] and Gopal et al[3] proved fixed point theorems for mappings satisfying some generalized contractive condition in Fuzzy Metric Space. The aim of our work is to prove common fixed point theorems for a sequence of mappings in a dislocated fuzzy metric space which extends and generalizes the results of Abbas et al[1], Cho et al [2], Gopal et al [3] and George and Khan [17].

2 Weakly compatible and horizontally weakly compatible mappings

Let \((X, d)\) be a metric space and \(\{(T_i, f_i)\}_{i \in N}\) be a collection of pairs of self valued mappings of \(X\). Let \(CP\{T_i, f_i\}\) denote the set of coincident points of the pair \((T_i, f_i)\) for each \(i\). If for each \(i\) there exists \(x_i \in X\), such that \(T_i x_i = f_i x_i = z\) for some \(z \in X\), then \(x_i\) is called a Horizontal Coincident Point of the pair \((T_i, f_i)\) in the collection \(\{(T_i, f_i)\}_{i \in N}\) and \(z\) is a Horizontal Point of Coincidence of the collection \(\{(T_i, f_i)\}_{i \in N}\). We denote by \(HCP\{T_i, f_i\}\) the set of all horizontal coincident points of the pair \((T_i, f_i)\). Note that \(HCP\{T_i, f_i\} \subseteq CP\{T_i, f_i\}\). We also denote by \(HPOC\{(T_i, f_i)\}\) the set of all horizontal points of coincidence of the family \(\{(T_i, f_i)\}\).

**Definition** If \(T_i f_i x = f_i T_i x\) for all \(x \in HCP\{T_i, f_i\}\), then we say that the pair \((T_i, f_i)\) is horizontally weakly compatible pair.

Clearly if the pair \((T_i, f_i)\) in the collection \(\{(T_i, f_i)\}_{i \in N}\) is weakly compatible then it is horizontally weakly compatible but the converse is not necessarily true.
Example 2.1 Let $X = [0, 1]$. Consider the family $\{(f_i, T_i)\}_{i=1,2}$ of self maps of $X$ given by:

$$f_1 x = x, \quad T_1 x = x^2$$
$$f_2 x = x - \frac{x^2}{2}, \quad T_2 x = x - \frac{x^3}{2}$$

We see that

$$f_1 0 = T_1 0 = 0 = f_2 0 = T_2 0$$
$$f_1 1 = T_1 1 = 1, \quad f_2 1 = T_2 1 = \frac{1}{2}.$$ 

Also $f_1 T_1 0 = T_1 f_1 0$ and $f_2 T_2 1 = T_2 f_2 1$. So the pairs $(f_1, T_1)$ and $(f_2, T_2)$ are horizontally weakly compatible pairs, although the pair $(f_2, T_2)$ is not weakly compatible.

Lemma 2.2 Let $(T_i, f_i)$ be a horizontally occasionally weakly compatible pair in the collection $\{(T_i, f_i)\}_{i \in N}$. If $\{(T_i, f_i)\}$ has a unique horizontal point of coincidence then the pair $(T_i, f_i)$ is horizontally weakly compatible.

Proof 1 Let $z$ be the unique horizontal common point of coincidence of $\{(T_i, f_i)\}$. Since the pair $(T_i, f_i)$ is horizontally occasionally weakly compatible, there exists $x \in \text{HCP}(T_i, f_i)$ such that $T_i(f_i x) = f_i(T_i x)$, i.e. $T_i z = f_i z$. Then for any $u \in \text{HCP}(T_i, f_i)$ we have $T_i f_i u = T_i z = f_i z = f_i T_i u$. Hence the pairs $(T_i, f_i)$ is horizontally weakly compatible.

3 Fuzzy Metric Space and Dislocated Fuzzy Metric Space

Definition 3.1 ([22]) A binary operation $\star : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $([0, 1], \star)$ is an abelian monoid with unit one such that, for all $a,b,c,d$ in $[0,1], a \star b \geq c \star d$ whenever $a \geq c$ and $b \geq d$.

Definition 3.2 Let $X$ be any non empty set, $\star$ be a continuous t-norm and $M : X^2 \times [0, \infty) \to [0, 1]$ be a fuzzy set. For all $x, y, z \in X$ and $t, s \in [0, \infty)$, consider the following conditions:

$FM1 : M(x, y, 0) = 0$
$FM2 : M(x, x, t) = 1$
$FM3 : M(x, y, t) = 1 \text{ and } M(y, x, t) = 1 \Rightarrow x = y$
$FM4 : M(x, y, t) = M(y, x, t)$
$FM5 : M(x, y, t + s) \geq M(x, z, t) \star M(z, y, s)$
$FM6 : M(x, y, .) : [0, \infty) \to [0, 1] \text{ is left continuous}$
$FM7 : M(x, y, .) : (0, \infty) \to [0, 1] \text{ is continuous}$
If $M$ satisfies conditions FM1 to FM6 then $(X, M, *)$ is called a Fuzzy Metric Space\cite{13}. If $M$ satisfies conditions FM1 and FM3 to FM6 then we say that $(X, M, *)$ is a Dislocated Fuzzy Metric Space in the sense of Kramosil and Michalek (in short $D_{KM}$FMSpace \cite{17}). If $M : X^2 \times (0, \infty) \to [0, 1]$ satisfies conditions FM1 and FM3 to FM5 and FM7 then we say that $(X, M, *)$ is a Dislocated Fuzzy Metric Space in the sense of George and Veeramani (in short $D_{GV}$FMSpace \cite{17}). By a Dislocated Fuzzy Metric Space (in short $DFM - Space$ ) we mean a $D_{KM}$FMSpace or $D_{GV}$FMSpace.

Example 3.3 Let $X = R$. Define $a * b = ab$ and $M(x,y) = \left[\exp\left(\frac{|x-y|+|x+y|}{4}\right)\right]^{-1}$ for all $(x, y) \in X \times X$, $t \in (0, \infty)$. Then $(X, M, *)$ is a $D_{GV}$FMSpace.

Definition 3.4 A sequence $\{x_n\}$ in a $DFM - Space$ $(X, M, *)$ converges to $x$ if and only if for each $\epsilon > 0$, there exist $n_0 \in N$ such that $M(x_n, x, t) > 1 - \epsilon$ for all $n \geq n_0$ and $t > 0$.

Proposition 3.5 Let $(X, M, *)$ be a $DFM - Space$ and $x_n$ be a sequence in $X$. If sequence $x_n$ converges to $x \in X$ then $M(x, x, t) = 1$ for all $t > 0$.

Proof We have $M(x, x, t) \geq M(x_n, x, t/2) * M(x_n, x, t/2)$ for all $n$. Taking the limit as $n \to \infty$ we have $M(x, x, t) \geq 1 * 1 = 1$.

Lemma 3.6 In a $DFM - Space$ $(X, M, *)$, if $M(x, y, qt) \geq M(x, y, t)$ for all $x, y \in X$, $t \geq 0$ and $q \in (0, 1)$ then $x = y$.

Let $A, B, S$ and $T$ be self mappings of $DFM - Space$ $(X, M, *)$. An element $z \in X$ is said to be a coincidence point of $A$ and $S$ if and only if $Az = Sz$. $z$ is said to be a common fixed point of $A$ and $S$ iff $Az = Sz = z$.

Definition 3.7 The pair $(A, S)$ is weakly compatible if and only if $M(ASz, SAz, t) = 1$ for all $z \in C(A, S)$ and $t \in [0, \infty)$.

Definition 3.8 The pair $(A, S)$ satisfy (EA) Property if and only if there exist a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ax_n = x$ for some $x \in X$.

Definition 3.9 The pairs of mappings $(A, S)$ and $(B, T)$ are said to be horizontally weakly compatible pairs if and only if $M(ASu, SAu, t) = 1$ and $M(BTv, TBv, t) = 1$ for all $t \in [0, \infty)$ whenever $u \in C(A, S)$, $v \in C(B, T)$ and $Au = Su = Bv = Tv$.

Example 3.10 Let $X = [0, 100]$ and $a * b = ab$. Let $M$ be the Fuzzy Metric induced by $d$, where $d(x, y) = |x - y| + \frac{|x+y|}{2}$, for all $x, y \in X$. Then $(X, M, *)$ is a Dislocated Fuzzy Metric space but not a fuzzy metric space. Define self maps $A, B, S, T$ on $X$ as follows:
Horizontal weak compatibility of mappings

\[ A_x = \begin{cases} 
0, & x = 0 \\
4, & x > 0 
\end{cases} \]

\[ B_x = \begin{cases} 
0, & x = 0 \\
10, & x > 0 
\end{cases} \]

\[ S_x = \begin{cases} 
0, & x = 0 \\
24, & x > 0 
\end{cases} \]

\[ T_x = \begin{cases} 
0, & x = 0 \\
24, & 0 < x \leq 50 \\
10, & x > 50 
\end{cases} \]

Let \( \phi_2(x_1, x_2, x_3, x_4, x_5) = \min(x_1, x_2, x_3, x_4, x_5) \). We have, \( C(A, S) = \{0\} \) and \( C(B, T) = \{0\} \cup (50, 100) \) and \( Au = Su = Bv = Tv \) only for \( u = v = 0 \). Clearly \( M(AS0, SA0, t) = 1 \), \( M(BT0, TB0, t) = 1 \), and so the pairs \( (A, S) \) and \( (B, T) \) are horizontally weakly compatible. However, \( 51 \in C(B, T) \) and \( M(BT51, TB51, t) \neq 1 \) and so the pair \( (B, T) \) is not weakly compatible.

**Definition 3.11** We say that a sequence of pairs of mappings \( (A_i, S_i) \) in a DFM - Space satisfy common (EA) property, if there exists sequences \( \{x^n\}_n \) in \( X \) such that for each \( i = 1, 2, \ldots, n \), \( \lim_{n \to \infty} A_i(x^n)_n = \lim_{n \to \infty} S_i(x^n)_n = x \) for some \( x \in X \).

## 4 Main Results:

Let \( \psi \) denote the family of all continuous functions \( F : [0, 1]^6 \to R \), satisfying the following conditions:

\[
F(u(kt), 1, u(t), 1, 1, u(t)) \geq 0 \text{ or } F(u(kt), 1, 1, u(t), u(t), 1) \geq 0 \text{ or } F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 0 \Rightarrow u(kt) > u(t)
\]

where \( u : [0, \infty] \to [0, 1] \) and \( t > 0 \).

Some Examples of functions of class \( \psi \) are as follows:

**Example 4.1** Define \( F : [0, 1]^6 \to R \) as \( F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi_{\{\min(t_2, t_3, t_4, t_5, t_6)\}} \)
where \( \phi : [0, 1] \to [0, 1] \) is a continuous function such that \( \phi(t) > t \) for \( 0 < t < 1 \)

**Example 4.2** Define \( F : [0, 1]^6 \to R \) as \( F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k_{\{\min(t_2, t_3, t_4, t_5, t_6)\}} \)
where \( k > 1 \)
Example 4.3 Define $F : [0,1]^6 \to R$ as $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - kt_2 - \min(t_3,t_4,t_5,t_6)$ where $k > 0$

Example 4.4 Define $F : [0,1]^6 \to R$ as $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^{1/3} - kt_2t_3t_4t_5t_6$ where $k > 1$

Example 4.5 Define $F : [0,1]^6 \to R$ as $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - at_2t_5 - bt_3 - ct_4t_6$ where $(a+c) > 1$ or $(b+c) > 1$

Example 4.6 Define $F : [0,1]^6 \to R$ as $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \phi(t_2,t_3,t_4,t_5,t_6)$ where $\phi : [0,1]^5 \to [0,1]$ is a non-decreasing function such that $\phi(t,t,t,t,t) > t$

Example 4.7 Define $F : [0,1]^6 \to R$ as $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \frac{1}{r(a_0(t_2,t_3)+(1-a_0(t_2,t_3)+t_4,t_5,t_6))}$

where $\phi_1 : [0,1]^2 \to [0,1]$ and $\phi_2 : [0,1]^5 \to [0,1]$ such that for $i = 1,2$

(a) $\phi_i$ is a non-decreasing function, for all variables
(b) $\phi_1(t,t) \geq t$, and $\phi_2(t,t,t,t,t) \geq t$ for all $t \in [0,\infty)$
(c) $\phi_i$ is continuous in all variables.

and $r : [0,\infty] \to [0,\infty]$ be a non-decreasing function such that $r(\theta) < \theta$ for all $\theta > 0$

Theorem 4.8 Let $(X,M,*)$ be a DFM – Space. $A,B$ and $\{f_i\}_{i \in N}$ be mappings from $X$ into itself such that

$$F(M(Ax,By,kt),M(f_{2i-1}x,f_{2i}y,t),M(f_{2i-1}x,By,t),M(f_{2i}y,At,t) ) \geq 0$$

for all $x,y \in X, \ t > 0, \ 0 < k < 1, \ i \in N$ and $F \in \psi$. Suppose the pairs $\{A,f_{2i}\}$ and $\{B,f_{2i-1}\}$ satisfy common (EA) property, $f_{2i}(X)$ and $f_{2i-1}(X)$ are closed subsets of $X$ and the pairs $\{A,f_{2i}\}$ and $\{B,f_{2i-1}\}$ are horizontally weakly compatible. Then $A,B$ and $\{f_i\}_{i \in N}$ have a unique common fixed point in $X$.

Proof Since the pairs $\{A,f_{2i}\}$ and $\{B,f_{2i-1}\}$ satisfies common (EA) property, there exist sequences $\{(x^n)\}$ in $X$ such that

$$\lim_{n \to \infty} A(x^{2n}) = \lim_{n \to \infty} f_{2i}(x^{2n}) = \lim_{n \to \infty} B(x^{2i-1})$$

for some $z \in X$ and each $i \in N$. Since $f_{2i}(X)$ is a closed subset of $X$, there exist $u_{2i} \in X$ such that $f_{2i}u_{2i} = z$ for each $i \in N$. From (0.2) we have,

$$F(M(Au_{2i},B(x^{2i-1}),kt),M(f_{2i}u_{2i},f_{2i-1}(x^{2i-1}),t),M(f_{2i}u_{2i},Au_{2i},t) ) \geq 0$$

As $n \to \infty$, we get
Horizontal weak compatibility of mappings

\[ F(M(Au_{2i}, z, kt), M(z, z, t), M(z, Au_{2i}, t), M(z, z, t), M(z, Au_{2i}, t)) \geq 0 \]

i.e.

\[ F(M(Au_{2i}, z, kt), 1, M(Au_{2i}, z, t), 1, 1, M(Au_{2i}, z, t)) \geq 0 \]

which implies

\[ M(Au_{2i}, z, kt) \geq M(Au_{2i}, z, t). \]

Hence \( Au_{2i} = z = f_{2i}u_{2i} \), i.e. \( C(A, f_{2i}) \neq \phi \).

Again since \( f_{2i-1}(X) \) is a closed subset of \( X \) there exist \( w_{2i-1} \in X \) such that \( f_{2i-1}w_{2i-1} = z \). Now we claim \( Bw_{2i-1} = f_{2i-1}w_{2i-1} \). From (0.2), we have,

\[
F(M(A(x^{2i})_{n}, Bw_{2i-1}, kt), M(f_{2i}(x^{2i})_{n}, f_{2i-1}w_{2i-1}, t), M(f_{2i}(x^{2i})_{n}, A(x^{2i})_{n}, t) \\
M(f_{2i-1}w_{2i-1}, Bw_{2i-1}, t), M(f_{2i}(x^{2i})_{n}, Bw_{2i-1}, t), M(f_{2i-1}w_{2i-1}, A(x^{2i})_{n}, t), ) \geq 0
\]

As \( n \to \infty \), we get

\[ F(M(z, Bw_{2i-1}, kt), M(z, z, t), M(z, Bw_{2i-1}, t), M(z, Bw_{2i-1}, t), M(z, z, t) \geq 0 \]

i.e.

\[ F(M(z, Bw_{2i-1}, kt), 1, 1, M(z, Bw_{2i-1}, t), M(z, Bw_{2i-1}, t), 1) \geq 0 \]

i.e. \( M(z, Bw_{2i-1}, kt) \geq M(z, Bw_{2i-1}, t) \) for all \( t > 0 \). Hence \( Bw_{2i-1} = z = f_{2i-1}w_{2i-1} \), i.e., \( C(B, f_{2i-1}) \neq \phi \). Thus we have \( Au_{2i} = f_{2i}u_{2i} = Bw_{2i-1} = f_{2i-1}w_{2i-1} = z \). Since the pairs (\( A, f_{2i} \)) and (\( B, f_{2i-1} \)) are horizontally weakly compatible, we have, \( M(Af_{2i}u_{2i}, f_{2i}Au_{2i}, t) = 1 \) and \( M(Bf_{2i-1}w_{2i-1}, f_{2i-1}Bw_{2i-1}, t) = 1 \), i.e \( M(Az, f_{2i}z, t) = 1 \) and \( M(Bz, f_{2i-1}z, t) = 1 \). Hence \( Az = f_{2i}z \) and \( Bz = f_{2i-1}z \). Now we claim \( Az = z \) From (0.2) we have,

\[ F(M(Az, Bw_{2i-1}, kt), M(f_{2i}z, f_{2i-1}w_{2i-1}, t), M(f_{2i}z, Az, t), \\
M(f_{2i-1}w_{2i-1}, Bw_{2i-1}, t), M(f_{2i}z, Bw_{2i-1}, t), M(f_{2i-1}w_{2i-1}, Az, t), ) \geq 0
\]

i.e.

\[ F(M(Az, z, kt), M(z, z, t), M(z, Az, t), M(z, z, t), M(z, Az, t)) \geq 0 \]

i.e.

\[ F(M(Az, z, kt), 1, M(Az, z, t), 1, 1, M(Az, z, t)) \geq 0 \]

which implies \( M(Az, z, kt) \geq M(Az, z, t) \) and so \( Az = z \). Thus we have \( Az = f_{2i}z = z \). Similarly \( Bz = f_{2i-1}z = z \) and so \( z \) is the common fixed point of \( A, B \) and \( \{ f_{i} \}_{i \in N} \). Now we prove that \( z \) is unique common fixed point. If not, let there exist another common fixed point \( p \) of \( A, B \) and \( \{ f_{i} \}_{i \in N} \) such that \( p \neq z \). From (0.2), we have
\[ F(M(Ap, Bz, kt), M(f_2p, f_2i-z, t), M(f_2p, Ap, t), M(f_2i-z, Bz, t), \\
M(f_2p, Bz, t), M(f_2i-z, Ap, t)) \geq 0 \]

i.e.

\[ F(M(Ap, z, kt), M(Ap, z, t), 1, 1, M(Ap, z, t), M(Ap, z, t)) \geq 0 \]

i.e. \( M(Ap, z, kt) \geq M(Ap, z, t) \) which implies \( M(p, z, kt) \geq M(p, z, t) \) and so \( p = z \). Thus \( z \) is the unique common fixed point of \( A, B \) and \( \{f_i\}_{i \in \mathbb{N}} \).

**Corollary 4.9** Let \((X, M, *)\) be a \(DFM - \)Space. \( A, B \) and \( \{f_i\}_{i=1,2\ldots n} \) be mappings from \( X \) into itself satisfying any of the following conditions for each \( i = 1, 2\ldots n \):

(i) \( M(Ax, By, kt) \geq \phi (\min \{M(f_{2i-1}x, f_{2i}y, t), M(f_{2i-1}x, Ax, t), M(f_{2i}y, By, t), M((f_{2i-1}x, By, t), \\
M(f_{2i}y, Ax, t)) \}) \) where \( \phi : [0, 1] \rightarrow [0, 1] \) is a continuous function such that \( \phi(t) > t \) for \( 0 < t < 1 \)

(ii) \( M(Ax, By, kt) \geq k(\min \{M(f_{2i-1}x, f_{2i}y, t), M(f_{2i-1}x, Ax, t), M(f_{2i}y, By, t), M((f_{2i-1}x, By, t), \\
M(f_{2i}y, Ax, t)) \}) \) and \( k > 1 \)

(iii) \( M(Ax, By, kt) \geq k(M(f_{2i-1}x, f_{2i}y, t) + \min \{M(f_{2i-1}x, Ax, t), M(f_{2i}y, By, t), M((f_{2i-1}x, By, t), \\
M(f_{2i}y, Ax, t)) \}) \) where \( k > 1 \)

(iv) \( M(Ax, By, kt) \geq kM(f_{2i-1}x, f_{2i}y, t)M(f_{2i-1}x, Ax, t)M(f_{2i}y, By, t)M((f_{2i-1}x, By, t))M(f_{2i}y, Ax, t) \)

where \( k > 1 \)

(v) \( M(Ax, By, kt) \geq a(M(f_{2i-1}x, f_{2i}y, t)M((f_{2i-1}x, By, t)) + b.M(f_{2i-1}x, Ax, t) \\
c(M(f_{2i}y, By, t)M(f_{2i}y, Ax, t)) \) where \( a + c > 1 \) or \( b + c > 1 \)

(vi) \( M(Ax, By, kt) \geq \phi (M(f_{2i-1}x, f_{2i}y, t), M(f_{2i-1}x, Ax, t), M(f_{2i}y, By, t), M((f_{2i-1}x, By, t), \\
M(f_{2i}y, Ax, t))) \) where \( \phi : [0, 1]^5 \rightarrow [0, 1] \) is a non-decreasing function such that \( \phi(t, t, t, t, t) > t \)

(vii) \( M(Ax, By, kt) \geq \\
\frac{1}{\alpha \phi_1(M(f_{2i-1}x, Ax, t), M(f_{2i}y, Ax, t)) + (1 - \alpha) \phi_2(M(f_{2i-1}x, f_{2i}y, t), M(Ax, By, t), M(f_{2i}y, Ax, t), M((f_{2i-1}x, By, t), M(f_{2i}y, Ax, t))^{-1}) + 1} \)

where \( \phi_1 : [0, 1]^2 \rightarrow [0, 1] \) and \( \phi_2 : [0, 1]^5 \rightarrow [0, 1] \) such that for \( i = 1, 2 \)

(a) \( \phi_i \) is a non-decreasing function, for all variables

(b) \( \phi_1(t, t) \geq t \), and \( \phi_2(t, t, t, t, t) \geq t \) for all \( t \in [0, \infty) \)

(c) \( \phi_i \) is continuous in all variables.

and \( r : [0, \infty] \rightarrow [0, \infty] \) be a non-decreasing function such that \( r(\theta) < \theta \) for all \( \theta > 0 \)

If the pairs \((A, f_{2i-1}), (B, f_{2i}), i = 1, 2\ldots n \) satisfy common \((EA)\) property and are horizontally weakly compatible pairs, then \( A, B \) and \( \{f_i\}_{i = 1, 2\ldots n} \) have a unique common fixed point in \( X \).
**Proof** The Proof of corollary 4.9 follows from Theorem (4.8) above and the Examples (4.1) to (4.7).

**Remark 4.10** Corollary 4.9 is a proper extension and generalisation of the results of Abbas et al [1], Cho et al [2], Gopal-et-al [3] and Imdad and Ali [11].

## 5 Application to integral type contraction

**Theorem 5.1** Let \( A, B, S, \) and \( T \) be four self mappings of a DFM–Space \((X, M, \ast)\). Assume that there exist a Lebesgue integrable function \( \varphi : R \to R \) and a function \( F : [0, 1]^6 \to R \) such that

\[
\int_0^\infty F(u(kt), u(t), 1, 1, u(t)) \varphi(s)ds \geq 0, \quad \int_0^\infty F(u(kt), 1, 1, u(t), 1) \varphi(s)ds \geq 0, \tag{5.1}
\]

where \( u : [0, \infty) \to [0, 1] \) and \( t > 0 \) implies \( u(kt) > u(t) \). Suppose the pairs \( \{A, S\} \) and \( \{B, T\} \) satisfy common \((EA)\) property, \( S(X) \) and \( T(X) \) are closed subsets of \( X \). If

\[
\int_0^\infty F(M(Ax, By, kt), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), M((Sx, By, t), M(Ty, Ax, t)) \varphi(s)ds \geq 0, \tag{5.3}
\]

\( \forall x, y \in X \) and \( t > 0 \), Then the pair \( \{A, S\} \) and \( \{B, T\} \) have a point of coincidence. Further \( A, B, S, \) and \( T \) have a unique common fixed point provided that both the pairs \( \{A, S\} \) and \( \{B, T\} \) are horizontally weakly compatible.

**Proof:** Since the pairs \( \{A, S\} \) and \( \{B, T\} \) satisfy common \((EA)\) property, there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z \) for some \( z \in X \). Since \( S(X) \) is a closed subset of \( X \), there exist \( u \in X \) such that \( Su = z \). Now we assert that \( Au = Su \). By (5.5) we have

\[
\int_0^\infty F(M(Au, By, kt), M(Su, Ty, t), M(Su, Au, t), M(Ty, By, t), M(Su, By, t), M(Ty, Au, t)) \varphi(s)ds \geq 0, \tag{5.4}
\]

On making \( n \to \infty \), it reduces to

\[
\int_0^\infty F(M(Au, z, kt), 1, M(z, Au, t), 1, 1, M(z, Au, t)) \varphi(s)ds \geq 0, \tag{5.5}
\]

which implies \( M(Au, z, kt) \geq M(Au, z, t) \) and so \( Au = z \).

Being \( T(X) \) a closed subset of \( X \), repeating the same argument, we can deduce that there exist a point \( w \in X \) such that \( Bw = Tw \). Since the pairs \( \{A, S\} \) and \( \{B, T\} \) are horizontally weakly compatible, we have, \( M(ASu, SAu, t) = 1 \) and \( M(BTw, TBw, t) = 1 \), i.e \( M(Az, S, t) = 1 \) and
\( M(Bz, Tz, t) = 1 \). Hence \( Az = Sz \) and \( Bz = Tz \). Now we claim \( Az = z \).

Using (0.5), with \( x = z \) and \( y = w \), we have

\[
\int_0^\varphi(s)ds \geq 0, \quad (5.6)
\]

That is

\[
\int_0^\varphi(s)ds \geq 0, \quad (5.7)
\]

which implies \( M(Az, z, kt) \geq M(Az, z, t) \) and so \( Az = z \).

Similarly we can prove that \( Bz = Tz = z \) and so \( z \) is the common fixed point of \( A, B, S, \) and \( T \).

Uniqueness of \( z \) is a consequence of condition (0.5).

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References


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