The b-Chromatic Number of Bistar Graph

Immanuel T. San Diego and Frederick S. Gella¹

Department of Mathematics, Institute of Arts and Sciences
Far Eastern University, Manila, Philippines

Abstract

A b-coloring is a proper vertex coloring of a graph such that each color class contains a vertex that has a neighbor in all other color classes and the b-chromatic number is the largest integer \( \varphi(G) \) for which a graph has a b-coloring with \( \varphi(G) \) colors. In this paper, we established the b-chromatic number of the central graph, middle graph, and total graph of the bistar graph \( B_{m,n} \).

Mathematics Subject Classification: 05C15, 05C76

Keywords: bistar graph, b-coloring, central graph, middle graph, total graph

1 Introduction

Throughout, we assume that a graph \( G \) is simple and connected. The b-coloring of \( G \) is the proper coloring of the vertices of \( G \) such that every color class has a representative adjacent to at least one vertex in each of the other color classes. The \( b \)-chromatic number \( \varphi(G) \) of graph \( G \) is the largest integer \( k \) such that \( G \) admits \( b \)-coloring.

The central graph of \( G \), denoted \( C(G) \) is obtained by subdividing each edge of \( G \) exactly once and joining all the non-adjacent vertices of \( G \). The middle graph of \( G \), denoted \( M(G) \), has vertex set \( V(G) \cup E(G) \) where two vertices \( x, y \) in the vertex set of \( M(G) \) are adjacent in \( M(G) \) in case one of the following holds:

¹Research is supported by the FEU-URC Research Grant, Philippines
1. \(x, y\) are in \(E(G)\) and \(x, y\) are adjacent in \(G\)

2. \(x\) is in \(V(G)\), \(y\) is in \(E(G)\), \(x, y\) are adjacent in \(G\).

The total graph of \(G\), denoted \(T(G)\), has vertex set \(V(G) \cup E(G)\) where two vertices \(x, y\) in the vertex set of \(T(G)\) are adjacent in \(T(G)\) in case one of the following holds

1. \(x, y\) are in \(V(G)\) and \(x\) is adjacent to \(y\) in \(G\)

2. \(x, y\) are in \(E(G)\) and \(x, y\) are adjacent to \(y\) in \(G\)

3. \(x\) is in \(V(G)\), \(y\) is in \(E(G)\) and \(x, y\) are adjacent in \(G\)

The bistar graph \(B_{m,n}\) is the graph obtained from \(K_2\) by joining \(m\) pendant edges to one end and \(n\) pendant edges to the other end of \(K_2\).

The concepts of \(b\)-coloring and \(b\)-chromatic number were introduced by Irving and Manlove in 1999, where they proved that determining \(\varphi(G)\) is NP-hard in general and polynomial for trees. An NP problem is the set of decision problems where the “yes”-instances can be accepted in polynomial time by a non-deterministic Turing machine. NP-hard is a class of problems which are at least as hard as the hardest problems in NP. Finding the \(b\)-chromatic number of general graphs belong to this class. Hence, many papers in \(b\)-coloring are focused on finding the \(b\)-chromatic number (or bound) of some classes of graphs.

The complexity of \(b\)-coloring were further studied in [2, 10]. The \(b\)-chromatic number of the Cartesian product of general graphs was studied in [8, 9]. In their series of papers, Effantin and Kheddouci studied power graphs of paths, cycles, trees and catterpillars.

Kouider and Zaker gave several bounds on \(\varphi(G)\) in terms of clique number and clique partition number of \(G\). Balakrishnan and Raj investigated the \(b\)-chromatic number of Mycielskians and vertex deleted subgraphs. Vijayalakshmi, Thilagavathi and Roopesh gave the exact value of the \(b\)-chromatic number of central graph, middle graph, total graph and line graph of star graphs. Klavzar and Jacovac, proved that the \(b\)-chromatic number of cubic graphs is 4 with the exception of Petersen graph, \(K_{3,3}\), prism over \(K_3\) and sporadic with 10 vertices. Vivin and Venkatachalam determined the \(b\)-chromatic number of corona graph of any graph \(G\) with path, cycle and complete graph, and they also generalizes the \(b\)-chromatic number on corona graph of any two graphs each on \(n\) vertices.
2 Preliminary Results

A $b$-coloring $\Gamma$ of $G$ is a partitioning $\Gamma = \{V_1, V_2, \ldots, V_k\}$ of the vertex set $V(G)$ of $G$ such that $V_i$ and $V_j$ have different colors whenever $i \neq j$ and each $V_i$ has a representative adjacent to at least one vertex in each of the other color classes. Each $V_i$ is called a color class.

A proper coloring of a graph $G$ must have at least $\omega(G)$ colors. Since the $b$-coloring is a proper coloring, $\phi(G) \geq \omega(G)$, where $\omega(G)$ is the clique number of $G$. The clique number of $G$ is the cardinality of the maximum clique in $G$. If $\Gamma = \{V_1, V_2, \ldots, V_k\}$ is a $b$-color classes partition of $G$, then $k$ must be at least the clique number of $G$. Also, for each $i \in \{1, 2, \ldots, k\}$, there exists $v_i \in V_i$ such that $\deg_G(v_i) \geq k$. This asserts that the $b$-chromatic number of $G$ cannot exceed the maximum degree of $G$ plus one. Thus, we have the following result.

**Theorem 2.1** Let $G$ be a graph of order $n$. Then $\omega(G) \leq \varphi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of $G$.

A direct consequence of the above theorem gives the $b$-chromatic number of the complete graph $K_n$. That is, for each $n \in \mathbb{N}$, $\varphi(K_n) = n$.

3 Central Graph of Bistar

Throughout, we assume that our graph is simple and connected. Moreover, we write $K_{1,m} = \{u\} + \overline{K}_m$, where $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $K_{1,n} = \{v\} + \overline{K}_n$, where $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. The bistar $B_{m,n}$ is obtained by adding an edge joining $u$ and $v$ of the stars $K_{1,m}$ and $K_{1,n}$.

The next result counts the number of vertices and edges of the bistar $B_{m,n}$.

**Theorem 3.1** Let $m, n \in \mathbb{N}$. Then $|V(B_{m,n})| = m + n + 2$ and $|E(B_{m,n})| = m + n + 1$.

**Proof**: The star $K_{1,m}$ has exactly $m + 1$ vertices while the star $K_{1,n}$ has exactly $n + 1$ vertices. Hence, the bistar has $m + n + 2$ vertices. On the other hand, $K_{1,m}$ has $m$ edges and $K_{1,n}$ has $n$ edges. Since the bistar is obtained by adding the edge $uv$, the size of the bistar $B_{m,n}$ is $m + n + 1$.  

The next theorem establishes the order and size of the central graph of the bistar $B_{m,n}$.

**Theorem 3.2** Let $m, n \in \mathbb{N}$. Then $|V(C(B_{m,n}))| = 2(m + n) + 1$ and $|E(C(B_{m,n}))| = \binom{m+n}{2} + 3(m + n) + 2$.
Proof: The order of the central graph of bistar is the sum of the size and order of the bistar. The result follows from Theorem 3.1. For the size of the central graph, each edge in each of the star is doubled. Hence, we have \(2(m+n)\) edges. In addition, the edge joining \(u\) and \(v\) is divided into two. This contributes two to the sum. Now, the vertices in \(K_m\) and the vertices of \(K_n\) are mutually non-adjacent in \(B_{m,n}\). Thus, they form a complete graph in the central graph of the bistar \(B_{m,n}\). This has \(\binom{m+n}{2}\) edges. Moreover, \(u\) is adjacent to every vertex of \(K_n\) and \(v\) is adjacent to every vertex of \(K_m\). Adding these gives the desired result.

Since every vertex added to \(B_{m,n}\) to construct the central graph of \(B_{m,n}\) is of order 2 and \(C(B_{m,n})\) has a 3-clique for \(m, n \geq 2\), the added vertices cannot constitute a color class. Hence, every color class of \(C(B_{m,n})\) must have a vertex of any of the stars. Thus, we have the following result.

**Theorem 3.3** Every color class in the \(b\)-coloring of the central graph of the bistar \(B_{m,n}\) must contain a vertex of a star.

The above theorem asserts that \(\varphi(B_{m,n}) \leq m + n + 2\).

In the proof of Theorem 3.2, a proper coloring of \(C(B_{m,n})\) must have at least \(m + n\) colors since \(C(B_{m,n})\) contains a clique of cardinality \(m + n\). On the other hand, \(u\) and \(v\) cannot be in different color classes. If we color \(u\) and \(v\) different from the \(m + n\) colors of the vertices of \(K_{m,n}\), we attain a \(b\)-coloring of \(C(B_{m,n})\). Hence, we have the following result.

**Theorem 3.4** Let \(m, n \in \mathbb{N}\). Then \(\varphi(C(B_{m,n})) = m + n + 1\).

4 Middle Graph of Bistar

The following result gives the order and size of the middle graph of the bistar graph \(B_{m,n}\).

**Theorem 4.1** Let \(m, n \in \mathbb{N}\). Then \(|V(M(B_{m,n}))| = 2(m + n) + 3\) and \(|E(M(B_{m,n}))| = \binom{m+2}{2} + \binom{n+1}{2} + n + m\).

Proof: This is clear.

Note that \(M(B_{m,n})\) has a clique number equal to \(\max\{m, n\} + 2\). Hence, \(\varphi(M(B_{m,n})) \geq \max\{m, n\} + 2\). Clearly, adding a color class cannot constitute a \(b\)-coloring. Hence, the following is immediate.

**Theorem 4.2** Let \(m, n \in \mathbb{N}\). Then \(\varphi(M(B_{m,n})) = \max\{m, n\} + 2\).
5 Total Graph of Bistar

The following result gives the order and size of $T(B_{m,n})$.

**Theorem 5.1** Let $m, n \in \mathbb{N}$. Then $|V(T(B_{m,n}))| = 2(m + n) + 3$ and $|E(T(B_{m,n}))| = \binom{m+2}{2} + \binom{n+1}{2} + 2(n + m) + 1$.

**Proof**: Note that the total graph is obtained from the middle graph by adding edges that joins the adjacent vertices of $G$. This is just the size of $B_{m,n}$ which is equal to $m + n + 1$. The result follows from Theorem 4.1.

The result here follows from the result in the middle graph of the bistar. Thus, we have the following result.

**Theorem 5.2** Let $m, n \in \mathbb{N}$. Then $\varphi(T(B_{m,n})) = \max\{m, n\} + 2$.

**Acknowledgements**. The authors thank the anonymous referees for their helpful suggestions and comments.

**References**


Received: July 5, 2014