Fractional Transformation Method for Constructing Solitary Wave Solutions to Some Nonlinear Fractional Partial Differential Equations

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Abstract

The aim of this paper is twofold: First, we derive both the factional KdV hierarchy and the fractional Burger hierarchy. Second, a fractional transform is conducted to convert nonlinear fractional partial differential equations (PDEs) into classical PDEs. Then, solitary ansatze methods will be used to obtain solitary wave solutions to the reduced PDEs. This new transformation has been tested to three different models of nonlinear fractional differential equations; the time- and space-fractional mKdV, time- and space-fractional Burger equation and time-fractional nonlinear biological population equation.

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1 Introduction and Discussion

Fractional order partial differential equations (PDEs), as generalizations of classical integer order PDE, are increasingly used to model problems in fluid flow, finance and other areas of application. Fractional derivatives provide an excellent tool for describing the heredity properties of various materials and process. For example, half-order derivatives and integrals prove to be more
useful for the formulation of certain electrochemical problems than the classical model [1]. Fractional differentiation and integration operators are also used for extensions of the diffusion and wave equations [2].

Over the past decade, many physicists and mathematicians have been intimately involved in finding and constructing novel and stable numerical analytical methods for solving fractional partial differential equations (FPDEs) of physical interests. Those numerical and analytical methods have included a variety of powerful techniques such that; finite difference method. Adomian decomposition method (ADM), Variational iteration method (VIM), Homotopy perturbation method (HPM), and others. A comprehensive survey reveals that ADM and VIM have been extensively used to solve FDEs since both generate immediate and visible terms of analytic solutions. Also, much work have been done by finite difference method because of its simplicity where it requires no linearization nor discretization.

Problems in fractional calculus are not only important but also quite challenging which usually involves hard mathematical solution techniques (see, e.g., [3]). Unfortunately, a general solution theory for almost each problem in this area has yet to be established. Each application venue has developed its own approaches and implementations. As a consequence, a single standard method for the problems in fractional calculus has not emerged. Therefore, finding reliable and efficient solution techniques along with fast implementation methods is a significantly important and active research area. Recently, He and Lee [4, 5, 6] suggested a fractional complex transform instrument that converts fractional derivatives into classical derivatives. They considered the following PDE:

\[ f(u, u_t^{\alpha}, u_x^{\beta}, u_{tt}^{2\alpha}, u_{xx}^{2\beta}, ...) = 0, \]  

where \( u_t^{\alpha} = \frac{\partial^{\alpha} u(t,x)}{\partial t^{\alpha}} \) denotes the modified Riemann-Liouville derivative; \( 0 < \alpha \leq 1 \) and \( 0 < \beta \leq 1 \). Then, the following transforms are introduced:

\[ s = \frac{p t^{\alpha}}{\Gamma(1 + \alpha)}, \]  

\[ X = \frac{q x^{\beta}}{\Gamma(1 + \beta)}, \]

where \( p \) and \( q \) are constants.

Using the above transforms, we can convert fractional derivatives into clas-
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Partial derivatives:
\[
\frac{\partial^\alpha u}{\partial t^\alpha} = p \frac{\partial u}{\partial s},
\]
\[
\frac{\partial^\beta u}{\partial x^\beta} = q \frac{\partial u}{\partial X}.
\]

We should mention here that the above equations are valid only for Jumarie modifications of Riemann-Liouville derivatives [7, 8, 9].

In [19], Joshi defined the whole KdV hierarchy in (1+1)-dimensions as follows:

Let \( P_{-1} = \frac{1}{2} \) and for \( n \geq 0 \), define the differential expressions \( P_n(u) \) recursively as
\[
\partial_x P_n(u) = (\partial_x^3 + 4u \partial_x + 2u_x)P_{n-1}(u).
\]

Then the KdV hierarchy is defined by
\[
u_t = \partial_x P_n(u), \; n \geq 1.
\]

For \( n = 0 \) the recursive formula in (6) gives \( P_0 = u \) and for \( n = 1 \), \( p_1 = u_{xx} + 6uu_x \). Therefore, the KdV hierarchy for \( n = 1 \) using definition (7) is given by
\[
u_t = u_{xxx} + 6uu_x.
\]

The first objective of this paper is to derive the whole fractional KdV hierarchy in (1+1)-dimensions. So, in a parallel manner to the above assumptions, let \( P_{-1} = \frac{1}{2} \) and for \( n \geq 0 \), define the fractional differential expressions \( P_n(u) \) recursively as
\[
\partial_x^\beta P_n(u) = (\partial_x^{3\beta} + 4u \partial_x^{\beta} + 2u_x^{\beta})P_{n-1}(u).
\]

Then the fractional Burgers hierarchy is defined by
\[
u_t^\alpha = \partial_x^\beta P_n(u), \; n \geq 1.
\]

For \( n = 1 \), equation (10) yields the following fractional KdV hierarch
\[
u_t^\alpha = u_x^{3\beta} + 6uu_x^{\beta}.
\]

Now, we move to derivations of the whole fractional Burgers hierarchy. Let \( P_{-1} = 1, \quad P_0(u) = u \), and for \( n \geq 1 \), define the fractional differential expressions \( P_n(u, u_x^{\beta}, ..., \partial_x^{\alpha \beta} u) \) recursively as
\[
P_n(u, ..., \partial_x^{\alpha \beta} u) = (u + \partial_x^{\beta})P_{n-1}(u, ..., \partial_x^{(n-1)\beta} u).
\]
Then the fractional Burgers hierarchy is defined by
\[ u_t^\alpha = \partial_x^\beta P_n(u, ..., \partial_x^{\beta n} u), n \geq 1. \]  
(13)
For more information about the classical Burger hierarchy, one can refer to [18].

The second purpose in this work, we implement some well known ansatze methods to study the solutions of the following fractional KdV equations:

1. The time- and space-fractional mKdV [21]
\[ D_t^\alpha u + \varepsilon u^m D_x^\beta u + \nu D_x^{3\beta} u = 0, \quad 0 < \alpha, \beta \leq 1, \quad t, x > 0, \]  
(14)
where \( \varepsilon, \nu \) are constants, \( m = 0, 1, 2 \) and \( \alpha, \beta \) are parameters describing the order of the fractional time and space derivatives, respectively. If \( m = 0, m = 1, m = 2 \), equation (14) becomes the linear fractional KdV, non-linear fractional KdV and fractional modified KdV, respectively. The function \( u(x, t) \) is assumed to be a casual function of time and space, i.e. vanishing for \( t < 0 \) and \( x < 0 \).

2. The time- and space-fractional Burger equation [24]
\[ D_t^\alpha u + u D_x^\beta u - \nu D_x^{2\beta} u = 0, \quad 0 < \alpha \leq 1, \quad t, x > 0. \]  
(15)
This equation demonstrates the coupling between diffusion and convection processes. The constant \( \nu \) defines the kinematic viscosity. If the viscosity \( \nu = 0 \), the equation is called inviscid Burgers equation.

Finally, we introduce and solve the time-fractional nonlinear biological population equation [22, 23]
\[ D_t^\alpha u = (u^2)_{xx} + (u^2)_{yy} + f(u), \quad 0 < \alpha \leq 1, \quad t > 0, \]  
(16)
where \( u \) denotes population density and \( f(u) = h u^{r_1} (1 - r_2 u^{r_3}) \) represents the population supply due to births and deaths, \( h, r_1, r_2, r_3 \) are real constants.

## 2 Solitary ansatze methods

In this section, we present briefly the sine-cosine function method, the tanh method and the exponential method in their systematized forms.

Suppose we are given a nonlinear evolution equation in the form of a partial differential equation PDE for a function \( u(t, x) \). First, we seek traveling wave solutions by taking \( u(x, t) = u(z), z = x - ct \), where \( c \) represents the
wave velocity of the traveling wave. Substitution into the PDE yields an ordinary differential equation (ODE) for \( u(z) \). The ordinary differential equation is then integrated as long as all terms contain derivatives, where the integration constants are considered as zeros. The resulting ODE is then solved by the following ansatze methods.

### 2.1 The sine-cosine Method

The sine-cosine method [11, 12, 13, 15] admits the use of ansatz

\[
\begin{align*}
  u(x,t) &= A \cos^B(\mu z), \quad |z| \leq \frac{\pi}{\mu} \\
  u(x,t) &= A \sin^B(\mu z), \quad |z| \leq \frac{\pi}{2\mu}
\end{align*}
\]

where \( A, \mu, c \) and \( B \) are parameters to be determined later. Substituting (17) or (18) into the reduced ODE gives a polynomial equation of cosine or sine terms. Balancing the exponents of the trigonometric functions cosine or sine, collecting all terms with same power in \( \cos^B(\mu z) \) or \( \sin^B(\mu z) \) and setting their coefficients to zero, we get a system of algebraic equations among the unknowns \( A, \mu \) and \( B \). The problem is now completely reduced to an algebraic one. Therefore, by determining \( A, \mu, c \) and \( B \) by algebraic calculations (or by using computerized symbolic calculations), the solutions proposed in (17) and in (18) follow immediately.

### 2.2 The tanh method

The tanh technique [10, 14, 16, 17] is based on the assumption that the traveling wave solutions can be expressed in terms of the tanh function. We therefore introduce a new independent variable

\[
Y = \tanh(\mu z).
\]

Then, the solution can be proposed a finite power series in \( Y \) in the form:

\[
u(\mu z) = S(Y) = \sum_{i=0}^{M} a_i Y^i,
\]

limiting them to solitary and shock wave profiles. The parameter \( M \) is a positive integer, in most cases, that will be determined by using a balance procedure, where by comparing the behavior of \( Y^i \) in the highest derivative against its counterpart within the nonlinear terms. With \( M \) determined, we collect all coefficients of powers of \( Y \) in the resulting equation where theses coefficients have to vanish, hence the coefficients \( a_i \) can be determined.
2.3 The exponential method

The exponential method [20] admits the use of the ansatz

\[ u(z) = \frac{1 + A_1 e^{\mu z} + A_2 e^{-\mu z}}{A_3 + A_4 e^{\mu z} + A_5 e^{-\mu z}}, \]

where \( A_1, A_2, A_3, A_4, A_5 \) and \( \mu \) are parameters that will be determined by collecting all coefficients of powers of \( e^{\mu z} \) in the resulting equation where these coefficients have to vanish.

3 Time- and space-fractional mKdV

In this section, we apply the sine-cosine method [10, 11, 12] to construct solutions of different types of the fractional mKdV.

3.1 Case I

We consider the following time-fractional mKdV equation that reads:

\[ D_t^\alpha u + 6u^2 u_x + u_{xxx} = 0, \quad 0 < \alpha \leq 1, \quad t > 0. \]  

(22)

Now, by means of the fractional transform, equation (22) becomes

\[ u_s + 6u^2 u_x + u_{xxx} = 0, \]

(23)

where \( u = u(s, x) \) with \( s = \frac{t^\alpha}{\Gamma(1+\alpha)} \). Then, we use the wave variable \( z = x - cs \) which reduce the PDE of (23) into the following ODE

\[ -cu + 2u^3 + u'' = 0, \]

(24)

where \( u = u(z) \) and the prime denotes the ordinary derivative.

Following the steps of applying the sine-cosine method to equation (24), the solutions of equation (22) are

\[ u(t, x) = \pm i \mu \sec \left( \mu \left( x + \mu^2 \frac{t^\alpha}{\Gamma(1+\alpha)} \right) \right), \]

(25)

and

\[ u(t, x) = \pm i \mu \csc \left( \mu \left( x + \mu^2 \frac{t^\alpha}{\Gamma(1+\alpha)} \right) \right). \]

(26)

Figure 1, shows profile solutions given in (25) for different values of the fractional order \( \alpha \).
3.2 Case II

We consider the following space-fractional mKdV equation that reads:

$$u_t + 6u^2D_x^\alpha u + D_x^{3\beta}u = 0, \quad 0 < \alpha \leq 1, \quad t, x > 0.$$ (27)

Using the fractional transform $X = \frac{x^\beta}{\Gamma(1+\beta)}$, equation (27) becomes

$$u_t + 6u^2u_X + u_{XXX} = 0,$$ (28)

where $u = u(t, X)$. Then, we use the wave variable $z = X - ct$ which reduce the PDE of (28) into the following ODE

$$-cu + 2u^3 + u'' = 0,$$ (29)

where $u = u(z)$ and the prime denotes the ordinary derivative.

Referring to the obtained results in (26), the solutions of equation (27) are

$$u_1(t, x) = \pm i \mu \sec \left( \mu \left( \frac{x^\beta}{\Gamma(1+\beta)} + \mu^2 t \right) \right),$$ (30)

and

$$u_2(t, x) = \pm i \mu \csc \left( \mu \left( \frac{x^\beta}{\Gamma(1+\beta)} + \mu^2 t \right) \right).$$ (31)

3.3 Case III

We consider the following time- and space-fractional mKdV equation that reads:

$$D_t^\alpha u + 6u^2D_x^\beta u + D_x^{3\beta}u = 0, \quad 0 < \alpha, \beta \leq 1, \quad t, x > 0.$$ (32)
Using the fractional transforms \(s = \frac{t^\alpha}{\Gamma(1+\alpha)}\) and \(X = \frac{x^\beta}{\Gamma(1+\beta)}\), equation (32) becomes
\[u_s + 6u^2u_X + u_{XXX} = 0,\] (33)
where \(u = u(t, X)\). Then, we use the wave variable \(z = X - cs\) which reduce the PDE of (33) into the same ODE as in (24) and (29). Therefore, the solutions of equation (32) are
\[u_1(t, x) = \pm i \mu \sec \left( \mu \left( \frac{x^\beta}{\Gamma(1+\beta)} + \mu^2 \frac{t^\alpha}{\Gamma(1+\alpha)} \right) \right),\] (34)
and
\[u_2(t, x) = \pm i \mu \csc \left( \mu \left( \frac{x^\beta}{\Gamma(1+\beta)} + \mu^2 \frac{t^\alpha}{\Gamma(1+\alpha)} \right) \right).\] (35)

Figure 2, shows plots of the obtained solutions in (34) for different values of \(\alpha\) and \(\beta\)

![Figure 2: Solutions of time- and space-fractional mKdV (32) (a) \(\alpha = 1\) and \(\beta = 1\), (b) \(\alpha = 0.5\) and \(\beta = 0.5\), (c) \(\alpha = 0.2\) and \(\beta = 0.8\), (d) \(\alpha = 0.8\) and \(\beta = 0.2\).](image-url)
4 Time- and space-fractional Burger equation

In this section, we apply the tanh method to solve the fractional Burger equations:

$$D_t^\alpha u + u D_x^\beta u - v D_x^{3\beta} u = 0, \quad 0 < \alpha \leq 1, \quad t, x > 0. \quad (36)$$

By means of the proposed fractional transform, equation (36) reduces to

$$u_s + uu_X - vu_{XX} = 0, \quad (37)$$

where $u = u(s, X)$. The wave variable $z = X - cs$ transforms the PDE (37) into the following ODE

$$-cu + (1/2)u^2 - vu' = 0, \quad (38)$$

where $u = u(z)$.

Following the steps of applying the tanh method to equation (38), the solution of equation (36) is

$$u(t, x) = A + A \tanh \left( \frac{A}{2v} \left( \frac{x^\beta}{\Gamma(1+\beta)} + A \frac{t^\alpha}{\Gamma(1+\alpha)} \right) \right). \quad (39)$$

5 Time-fractional nonlinear biological population equation

In this section, we study the time-fractional nonlinear biological population equation:

$$D_t^\alpha u = (u^2)_{xx} + (u^2)_{yy} + hu, \quad 0 < \alpha \leq 1, \quad t > 0, \quad (40)$$

for the case $r_1 = 1, r_2 = 0$

The fractional transform $s = \frac{t^\alpha}{\Gamma(1+\alpha)}$ reduces (40) to

$$u_s = (u^2)_{xx} + (u^2)_{yy} + hu \quad (41)$$

where $u = u(s, x, y)$. The wave variable $z = ax + by - cs$ transforms the PDE (41) into the following ODE

$$-cu' = (a^2 + b^2)(u^2)' + hu, \quad (42)$$

where $u = u(z)$. 
Following the steps of applying the exp-function method to equation (42), the solutions of equation (40) are

\[ u_1(t, x) = \frac{e^{-(ax+by-h\frac{t^\alpha}{\Gamma(1+\alpha)})}}{A_4}, \quad (43) \]

and

\[ u_2(t, x) = \frac{1 + \frac{A_3 e^{-(ax+by-h\frac{t^\alpha}{\Gamma(1+\alpha)})}}{A_4}}{A_3 + A_4 e^{(ax+by-h\frac{t^\alpha}{\Gamma(1+\alpha)})}}, \quad (44) \]

where \( A_3, A_4 \) are free constants. Figure 3, shows profile solutions given in (43) for different values of the fractional order \( \alpha \)

![Profile solutions](image)

Figure 3: Profile solutions given in equation (43) of the time-fractional biological population equation when \( x = y = 1, \ a = b = 1, \ h = 1, \ A_4 = 1 \) for different values of \( \alpha \) and \( 0 \leq t \leq 5 \)

### 6 Conclusion

In general, there exist no method that yields solitary solutions for fractional solitary wave equations. But, a fractional wave transform is adopted in this paper to convert such equations into classical partial differential equations. We succeeded in extracting solitary solutions for some fractional PDEs by applying different ansatze methods.

### References


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