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Demand and Equilibrium in Risk Exchange Economies

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Abstract

In this paper, we study demand correspondences and equilibrium in **risk exchange economies**. In these exchange economies, a finite number of regulators exhange their financial positions in order to minimize the capital requirements needed so that the final financial position to be more safe than the initial one, under some strictly positive price. The capital requirement functionals which represent the notion of safety for any regulator, are actually coherent risk measures.

Mathematics Subject Classification: 46B40, 91B30

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1 Introduction: Previous related work and paper's motivation

FIrst of all we consider a filtered probability space (stochastic base) $(\Omega, \mathbf{F}, \mathcal{F}, \mu)$, $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$, which satisfies the usual conditions of right-continuity and completeness. The seminal work about demand correspondences in infinite-dimensional spaces is the one of I.A. Polyrakis, [12]. In [12, Th.4], I.A.Polyrakis establishes the relation between the non-emptiness of the values of the demand correspondence for a typical exchange economy with the boundedness of the bases defined on a cone P in E, if $\langle E, F\rangle$ is an *ordered dual system*. For the

notion of the *ordered dual system*, see also [11]. In the present paper we study the demand correspondence in exchange economies, in which regulators exchange portfolios which are exposed to credit risk, in order to reduce it in a certain time -horizon. The static optimization problem that each regulator faces (minimize the risk of the new portfolio, given the valuation of the initial one) is solved without reference to the boundedness of the regulators' budget sets (which are actually bases of the cone E_+). This is achieved through the robust representation form of the risk preferences of the regulators, arising from the robust (dual) repesentation of their individual risk measures. This provides a sup-inf form on the optimal value of any risk-minimization problem, which may be solved via the saddle -point Theorem [4, p.10]. This is a specific approach on the demand correspondence existence problem, corresponding to the form of the risk utility arising here from a coherent risk measure which admits a robust representation. The importance of the equilibrium result we deduce here, is that it is independent from the boundedness of the bases defined on the positive cone of the space of the financial positions in the one -period uncertainty model, which we finally consider. It uses the idea of existence of demand correspondences in infinite-dimensional exchange economies developed extensively in [12], but the fact that this idea does not make use of the properties of bases both with the facts that it is applied on a risk-financial model, where a general equilibrium existence result is deduced, make it a contribution in the subject of demand correspondences and general equilibrium in exchange economies with financial markets. If $(T_n^i)_{n\in\mathbf{N}}$ is a sequence of stopping times related to the stochastic base we proposed, such that the price process $(S_t^i)_{t\in[0,T_n]}$ of the traded asset $i = 0, 1, 2, ..., d$ (see also below) to be such that $|S_t^i(\omega)| \leq a_n, \mu-a.e.$ in this case, where $a_n \to \infty$, then the process $S_t^i)_{t\in[0,T]}$ is called *locally bounded*. The stochastic -base frame we use is the one of the locally bounded price processes to assure that all $L^p(\Omega, \mathcal{F}_T, \mu), 1 \leq p \leq \infty$ may be considered as time-period T commodity spaces. To achieve this, we use the version of the Fundamental Theorem of Asset Pricing, which is considered by F. Delbaen and W. Schachermayer in [5, Th.1.1], but essentially in the [5, Cor.1.2]. A question which arises is 'How adequate is this general equilibrium model, in the case of eceonomic crises?' The answer to this important question may be given through trying to understand 'what an economic crisis is' from the aspect of the distributional properties of $S_t^i, t \in [0, T]$, $i = 1, 2, ..., d$. We recall of the notion of the *moment index* of a random variable $X_t \in L^0(\Omega, \mathcal{F}_t, \mu)$, which is equal to $I(X_t) = \sup\{p > 0|\mathbf{E}_\mu(|X_t^p|) < \infty\}$. Also, X_t is a heavy -tailed random variable if and only if $\mathbf{E}_\mu(e^{rX_t}) = \infty$ for any $r > 0$, which implies $I(X_t) < \infty$. Now we may characterize the traded risky assets of the market, whose price processes are $(S_t^i, t \in [0, T]), i = 1, 2, ..., d$ as a signal of the kind of the behaviour of the financial economy as a whole.

Definition 1.1 *(i) A market* $(S^1, S^2, ..., S^d), S^i = (S_t^i, t \in [0, T]), i =$

$1, 2, ..., d$ ***strictly corresponds*** *to a possible financial economy crisis, if and only if* $I(S_t^i) = 1$*, for any* $i = 1, 2, ..., d$ *and any* $t \in [0, T]$*.*

(*ii*) *A market* $(S^1, S^2, ..., S^d), S^i = (S_t^i, t \in [0, T]), i = 1, 2, ..., d$ ***weakly corresponds*** *to a possible financial economy crisis, if and only if* $I(S_t^i) < \infty$*, for any* $i = 1, 2, ..., d$ *and any* $t \in [0, T]$*.*

(*iii*) *A market* $(S^1, S^2, ..., S^d), S^i = (S_t^i, t \in [0, T]), i = 1, 2, ..., d$ ***partially corresponds*** *to a possible financial economy crisis, if and only if* $I(S_t^i) < \infty$*, for any* $i = 1, 2, ..., d$ *and any* t *with* $T \geq t \geq \tau$*, where* τ *is a stopping time with respect to the relevant stochastic base and for any* t *with* $0 \leq t \leq \tau$*,* $I(S_t^i) = \infty$ *holds.*

(*iv*) *A market* $(S^1, S^2, ..., S^d), S^i = (S_t^i, t \in [0, T]), i = 1, 2, ..., d$ ***partially does not correspond*** *to a possible financial economy crisis, if and only if* $I(S_t^i) = \infty$*, for any* $i = 1, 2, ..., d$ *and any* t *with* $T \geq t \geq \tau$*, where* τ *is a stopping time with respect to the relevant stochastic base and for any* t *with* $0 \leq t \leq \tau$*,* $I(S_t^i) < \infty$ *holds.*

(*vi*) *A market* $(S^1, S^2, ..., S^d), S^i = (S_t^i, t \in [0, T]), i = 1, 2, ..., d$ ***weakly does not correspond*** *to a possible financial economy crisis, if and only if* $I(S_t^i) = \infty$*, for any* $i = 1, 2, ..., d$ *and any* $t \in [0, T]$*.*

Under this Definition, we conclude that L^p spaces through moment index provide a frame of adequacy for this general equilibrium model -even for the situations related to the crises of financial economy.

A motivation for the use of L^p spaces is also provided by [5, Th.2.3] in which the variation norms of the maximum processes arising from the component processes of a semi-martingale, are controlled by the variation norms of the maximum process of the semi-martingale itself. [5, Cor.2.3] expresses the same reult for stoppped component maximum process, under a stopping -time. these results are deduced however for all $p \in (1, \infty]$ and they are actually related to the previous set of definitions. However, we deduce the existence of ES_a -risk exchange equilibrium for the case of $p = 1$, as well.

2 Risk Exchange Economies: model setup

Consider an *ordered normed linear space* E, which is a linear subspace of the *vector lattice* of real-valued random variables $L^0(\Omega, \mathcal{F}, \mu)$ over a probability space $(\Omega, \mathcal{F}, \mu)$. In a more general fashion, we consider an *ordered dual system* $\langle E, E^* \rangle$, where the topological dual E^* of E is in general a subspace of the *order dual* E^o. We also consider a time -horizon, which is $[0, T]$, $T > 0$. The space E represents the set of the *financial positions*. A rational property of

E is supposed to be the **Risk Diversification Property**, which is actually the well-known *Riesz Decomposition Property* (RDP): This property indicates that a position less than the sum of two other positions, may be expressed as the sum of two positions, each of ones is less than the previously mentioned positions. The RDP implies that the *order dual* E^o of E is a *vector lattice*, see [1, Th.8.24]. If E is moreover a Banach lattice, then $E^* = E^o$, see [1, Th.9.11]. In the rest of the paper, we suppose that $E = L^{1+\varepsilon}$, $\varepsilon \geq 0$. Hence, the requirement that E has to be a Banach lattice is satisfied. E is endowed by $\sigma(E, E^*)$-topology and E^* by $\sigma(E^*, E)$-topology. We also consider a finite set $\{1, 2, ..., I\}$ of *regulators*. Each regulator has her own *initial* financial position $e_i \in E_+$, $i = 1, 2, ..., I$. This position represents the situation of her total assets' portfolio at time -period 0. Such a financial position e_i is actually a random variable, due to the fact that the value of the total assets' portfolio of some financial institution is subject to change according to the state related to the credit risk $\omega \in \Omega$, which becomes true. Without loss of generality, we may suppose that the initial (corresponding to the time -period 0) financial positions of the regulators $i = 1, 2, ..., I$ lie in the *positive cone* E_+ of E. By the notion *financial institution* we denote a bank, a fund etc. The notion of the *regulator* corresponds to the one of risk manager. We suppose that the positive cone E_+ is weakly closed **(Condition (A))**. The *total initial position* of the set of the regulators $\{1, 2, ..., I\}$ is

$$e = \sum_{i=1}^{I} e_i.$$

The set of the *risk allocations* with respect to the total position e is

$$A_e = \{\mathbf{x} = (x_1, x_2, ..., x_I) \in (E_+)^I | \sum_{i=1}^{I} x_i = \sum_{i=1}^{I} e_i = e\}.$$

The (coherent) *risk measure* of each regulator is $\rho_i : E_+ \to \mathbf{R}$, $i = 1, 2, ..., I$ and its *robust representation* is

$$\rho_i(x) = \sup_{\pi_i \in D_i} \pi_i(-x), x \in E_+,$$

where $\pi_i \in E^*$ and D_i is a $\sigma(E^*, E)$-compact, convex set of E^*, which may be supposed to lie in E^*_+. We also assume that the elements of D_i are strictly positive functionals of E_+ and $\cap_{i=1}^{I} D_i = D_1 \neq \emptyset$ **(Condition (B))**. We recall that $f \in E^*$ is a strictly positive functional of E_+, if and only if $f(x) > 0, x \in E_+ \setminus \{0\}$. The *utility function* of each regulator is

$$u_i : E_+ \to \mathbf{R}, u_i(x) = -\rho_i(x),\ i = 1, 2, ..., I.$$

The domain of the utility function is E_+, since the regulators exchange financial positions that *clear the risk markets* (they select some $\mathbf{x} \in A_e$ at time -period T). A *risk exchange economy* is a triplet

$$(E_+, \{1, 2, ..., I\}, \{(e_i, \rho_i), i = 1, 2, ..., I\}).$$

The model of the financial market relies on the filtered probability space $(\Omega, \mathbf{F}, \mathcal{F}, \mu)$, $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$, which satisfies the usual conditions of right-continuity and completeness. There are $d + 1$ traded assets ($d \in \mathbf{N}$), whose discounted price processes are modelled by an $\mathbf{R}^{d+1}$-valued locally bounded semi-martingale $S = (S(0); S_t)_{t\in[0,T]} = (S_0; S_t^1, ...S_t^d)_{t\in[0,T]}$. The asset $k = 0$ plays the role of a numeraire security or a discount factor. We may set $S(0) := 1, \mu - a.s.$. We may also suppose that the gains from traiding at the time-period T are equal to $x_i - e_i$ for each $i = 1, ..., I$. Each regulator is considered to follow a trading strategy $\theta_i = (\theta_i^0, \theta_i^1, ..., \theta_i^d), i = 1, 2, ..., I$, where $\theta_i^k = (\theta_{i,t}^k), t \in [0, T], k = 1, 2, ..., d, i = 1, 2, ..., I$. Hence, we may write $x_i - e_i = (S \cdot \theta^i)_T$, where

$$(S \cdot \theta^i)_T = \int_0^T \theta_t dS_t = \sum_{k=0}^{d} \int_{[0,T]} \theta_{i,t}^k dS_t^k.$$

The set of *admissible trading strategies* Θ is defined as follows: $\{\theta = (\theta_t)_{t\in[0,T]}|$ is $\mathbf{F}$-adapted, $(S \cdot \theta)_T \in L^{1+\varepsilon}, \varepsilon > 0\}$. The actual meaning of *risk market clearing* comes from the form of the *budget set*

$$B(e_i) = \{x_i \in L_+^{1+\varepsilon} | x_i - e_i = (S \cdot \theta)_T, \theta \in \Theta\}.$$

3 The risk -demand correspondence

The problem that each regulator faces is to maximize her utility (actually to minimize her risk measure), with respect to some price functional $f \in E^*$ on the financial positions. This price functional is a *strictly positive functional* of E_+. We assume that $f \in D_i, i = 1, 2, ..., I$, **(Condition (C))**. Since $e_i \neq 0$, we have that $f(e_i) > 0, i = 1, 2, ..., I$. The *risk -demand set* of each regulator under the price f is the following:

$$x(\rho_i, e_i, f) = \{x \in E_+ | \rho_i(x) = \inf\{\rho_i(y) | y \in B(f, e_i)\}\},$$

where $B(f, e_i)$ is the base of the cone E_+

$$\{x \in E_+ | f(x) = f(e_i)\}.$$

In case where $E = L^{1+\varepsilon}$, the equality $B(f, e_i) = B(e_i)$ is assured by the First Fundamental Theorem of Asset Pricing.

Proposition 3.1 *Each risk measure ρ_i is $\sigma(E, E^*)$ -lower semicontinuous, $i = 1, 2..., I$.*

Proof: $\rho_i(x) = \sup_{\pi_i \in D_i} \pi_i(-x)$, hence for each $i = 1, 2, ..., I$ ρ_i is the pointwise supremum of $\sigma(E, E^*)$- lower semicontinuous functionals, since $\rho_i(x) = \sup_{f_i \in G_i} f_i(x)$, where $f_i(x) = \pi_i(-x), x \in E$, $G_i = -D_i, i = 1, 2, ..., I$, see also [1, Lem.2.41]. We also have in mind that the dual space E^* of E is also the weakly dual space of E (see [1, Pr.5.93]).

Proposition 3.2 *Each risk measure ρ_i attains a minimum value on any base $B(f, e_i)$ of the cone E_+, $i = 1, 2, ..., I$, which implies that the risk -demand correspondence is well-defined for each regulator.*

Proof: Arises directly from the min-max Theorem mentioned in [4, p.10]. We recall its statement. *Let K be a compact, convex subset of a locally convex space Y. Let L be a convex subset of an arbitrary vector space M. Suppose that u is a bilinear function $h : M \times Y \to \boldsymbol{R}$. For each $l \in L$, we suppose that the partial (linear) function $h(l, .)$ is continuous on Y. Then we have that*

$$\inf_{l \in L} \sup_{k \in K} h(l, k) = \sup_{k \in K} \inf_{l \in L} h(l, k).$$

By this way, the existence of a minimum point x_i^* of ρ_i is assured through a corresponding saddle -point $(x_i^*, \pi_i^*) \in B(f, e_i) \times D_i$, see also [3, Pr.3.1]. Then, $x_i^* \in x(\rho_i, e_i, f), i = 1, 2, ..., I$.

We also define the set

$$m(\rho_i, e_i, f) = \{\pi \in D_i | \pi(-x) = \rho_i(x), x \in x(\rho_i, e_i, f)\}, i = 1, 2, ..., I.$$

These price sets do define another correspondence $m(\rho_i, e_i, f) : E \to 2^{E^*}$, or more specifically $m(\rho_i, e_i, f) : x(\rho_i, e_i, f) \to 2^{D_i}$. For the sake of simplicity we denote $m(\rho_i, e_i, f)(x) = m_i(x)$ for any $x \in x(\rho_i, e_i, f)$.

Proposition 3.3 *The price sets $m_i(x)$ are convex in E^*, for any $i = 1, 2, ..., I$, $x \in x(\rho_i, e_i, f)$.*

Proof: If $\pi_1, \pi_2 \in m_i(x)$ and $\lambda \in (0, 1)$, then for any $\pi_1, \pi_2 \in m_i(x)$, we get that $\pi_1(-x) = \rho_i(x), \pi_2(-x) = \rho_i(x)$. Hence $(\lambda \cdot \pi_1 + (1 - \lambda) \cdot \pi_2)(-x) = \rho_i(x)$. Hence the set $m_i(x)$ is convex.

Proposition 3.4 *The price sets $m_i(x)$ are $\sigma(E^*, E)$ -closed in E^*, for any $i = 1, 2, ..., I$.*

Proof: We will apply the Berge's Maximum Theorem. We notice that $\rho_i(x) = \max_{\pi \in D_i} \pi(-x)$ for any $x \in x(\rho_i, e_i, f)$, while the correspondence $\phi : E \to 2^{E^*}$, where $\phi(x) = D_i$ is weak-to-weak star continuous. The function $f : Gr\phi \to \mathbf{R}$, where $f(x, \pi) = \pi(-x)$ is continuous with respect to the induced product topology on the graph of ϕ, in which E and E^* are considered to be endowed by $\sigma(E, E^*)$ and $\sigma(E^*, E)$- topologies, respectively. Then by the Berge's Maximum Theorem, the 'argmax' correspondence m_i is upper hemicontinuous. Namely, for a net $(\pi_a)_{a \in A} \subseteq D_i$ such that $\pi_a \in m_i(x_a), x_a \in x(\rho_i, e_i, f)$ such that $x_a \overset{\sigma(E,E^*)}{\to} x$, there is some $\pi \in D_i$ such that $\pi_a \overset{\sigma(E^*,E)}{\to} \pi$ and $\pi \in m_i(x)$. For the specific case, we pose $x_a = x$ for some $x \in x(\rho_i, e_i, f)$.

Proposition 3.5 *The correspondence m_i is lower hemicontinuous with respect to the equivalent topologies, for any $i = 1, 2, ..., I$.*

Proof: Consider some net $(x_a)_{a \in A} \subseteq E_+$ such that $x_a \overset{\sigma(E,E^*)}{\to} x$. Then, for any $\pi \in m_i(x)$ since $\pi \in D_i$, there is some net $(\pi_\lambda)_{\lambda \in \Lambda}$, such that $\pi_\lambda \overset{\sigma(E^*,E)}{\to} \pi$. We define the following map (given that $m_i(x_{a_\lambda}) \neq \emptyset, \lambda \in \Lambda$) :

$$f : \Lambda \to \Pi_{\lambda \in \Lambda} m_i(x_{a_\lambda}) :$$

$$f(\lambda) = \{\pi_\lambda \in m_i(x_{a_\lambda}) | \pi_\lambda \overset{\sigma(E^*,E)}{\to} \pi,$$
$$\pi \in m_i(x), x_a \overset{\sigma(E,E^*)}{\to} x\}.$$

The index set $\{a_\lambda, \lambda \in \Lambda\}$ selected by this way is a directed subset of A. Hence, $x_{a_\lambda}, \lambda \in \Lambda$ is a subnet of $x_a, a \in A$. π_{a_λ} are selected such that $\pi_{a_\lambda} \in m_i(x_{a_\lambda}) \neq \emptyset$. Hence m_i is lower hemicontinuous.

Corollary 3.6 *The correspondence $m_i, i = 1, 2, ..., I$ is continuous.*

Proof: The correspondence $m_i, i = 1, 2, ..., I$ is both upper hemicontinuous and lower hemicontinuous, with respect to the equivalent topologies.

We notice that the optimal value of the maximization problem of each regulator $i = 1, 2, ..., I$ takes the form:

$$\max\{u_i(x) | x \in B(f, e_i)\} = -\min\{\rho_i(x) | x \in B(f, e_i)\} =$$
$$= -\min\{\pi(-x) | \pi \in m_i(x)\}, x \in B(f, e_i)\} =$$
$$= -\min\{-\pi(x) | \pi \in m_i(x)\}, x \in B(f, e_i)\} =$$
$$\max\{\pi(x) | \pi \in m_i(x)\}, x \in B(f, e_i)\}.$$

Proposition 3.7 *The 'argmax' correspondence of the maximization problem of each regulator $i = 1, 2, ..., I$ is upper hemicontinuous (namely, the risk-demand correspondence is upper hemicontinuous), with respect to the equivalent topologies.*

Proof: We will apply the Berge's Maximum Theorem. We notice that $v_i(x) = \max_{\pi \in m_i(x)} \pi(x)$ for any $x \in B(f, e_i)$, while the correspondence $\phi : E \to 2^{E^*}$, where $\phi(x) = m_i(x), x \in B(f, e_i)$ is weak-to-weak star continuous. The function $f : Gr\phi \to \mathbf{R}$, where $f(x, \pi) = \pi(x)$ is continuous with respect to the induced product topology on the graph of ϕ, in which E and E^* are considered to be endowed by $\sigma(E, E^*)$ and $\sigma(E^*, E)$- topologies, respectively. Then by the Berge's Maximum Theorem, the 'argmax' correspondence $x(\rho_i, e_i, f) = \{x \in B(f, e_i)|\rho_i(x) =$ is upper hemicontinuous.

Proposition 3.8 *The risk -demand correspondence of each regulator $i = 1, 2, ..., I$ is upper hemicontinuous and it has non-empty, compact, convex values.*

Proof: The fact that the values of the risk -demand correspondence of each regulator $i = 1, 2, ..., I$ ae non-empty and $\sigma(E, E^*)$-compact comes from the conclusion of the Berge's Maximum Theorem. The convexity of the values comes from the form of the optimal value

$$\max\{\pi(x)|, \pi \in m_i(x)\}, x \in B(f, e_i)\}.$$

We assume that $x_1, x_2 \in x(\rho_i, f, e_i)$. Given some $\lambda \in (0, 1)$, there are some $\pi_1 \in m_i(x_1), \pi_2 \in m_i(x_2)$, such that $\pi_1(x_1 = \pi_2(x_2) = m$, where m is the mimimum value of ρ_i over $B(f, e_i)$. Then, $\lambda x_1 + (1 - \lambda)x_2 \in x(\rho_i, f, e_i)$, since $m_i(x_1) = m_i(x_2)$. The same can be deduced for a finite convex combination.

Proposition 3.9 *The excess-demand corespondence of a risk exchange economy, which satisfies the Conditions (A)-(C) is upper hemicontinuous and it has non-empty, compact values.*

Proof: The finite sum of correspondences being non-empty, compact -valued with respect to the equivalent topologies and also upper hemicontinuous preserves these properties.

Definition 3.10 *A Risk Exchange Equilibirium $(\boldsymbol{x}, \pi)$ is consisted by an allocation $\boldsymbol{x} \in A_e$ and a pricing functional $\pi \in \cap_{i=1}^{I} D_i$, such that $x_i \in x(\rho_i, e_i, \pi)$.*

As it is well-known from [10, Th.4.1], Expected Shortfall measure admits the following dual representation

$$ES_{a,\varepsilon}(x) := \sup_{Q\in\mathcal{Z}_{a,\varepsilon}} \mathbf{E}_Q(-x)\,,$$

which may be defined on $L^{1+\varepsilon}$ for any $\varepsilon \geq 0$, where $a \in (0,\,1)$ and

$$\mathcal{Z}_{a,\varepsilon} = \left\{ Q \ll \mu \,\Big|\, 0 \leq \frac{dQ}{d\mu} \leq \frac{1}{a},\ \mu - a.s.\,,\ \frac{dQ}{d\mu} \in L^{1+1/\varepsilon}(\Omega, \mathcal{F}, \mu) \right\}.$$

Definition 3.11 *A Risk Exchange Economy in which $\rho_i = ES_{a_i}$ is called Expected Shortfall Risk Exchange Economy.*

We assume that the regulators calculate the adequate capital in order to secure their positions, according to Expected Shortfall measures ES_{a_i}, which are associated with the financial market of $d+1$ assets. For this reason, we assume that the probability measures in $\mathcal{Z}_{a,\varepsilon}$ are actually Equivalent Martingale Measures, from the First Fundamental Theorem of Asset Pricing -see [5, Th.1.1] (**Condition (D)**). At this point we have to mention the following

Definition 3.12 *A subset A of a locally convex space L is called* ***almost convex*** *if for any neighborhood V of 0 and any $\{d_1, d_2, ..., d_n\} \subseteq A$, there exist $\{w_1, w_2, ..., w_n\} \subseteq A$, such that $d_i - w_i \in V$, $i = 1, 2, ..., n$ and*

$$co\{w_1, w_2, ..., w_n\} \subseteq A.$$

We also remind the statement of [8, Th.1]: *Let K be a nonvoid compact subset of a separated locally convex space L, and $G : K \to 2^K$ be an upper hemicontinuous multifunction (correspondence), such that $G(x)$ is closed for all $x \in K$ and convex for all x in some dense almost convex subset A of K. Then G has a fixed point.*

Theorem 3.13 *Under Conditions (A), (B), (C), any Expected Shortfall Risk Exchange Economy has a Risk Exchange Equilibirium in $L^{1+\varepsilon} = E$.*

Proof: D_1 in this case is an order interval in $L^{1+\frac{1}{\varepsilon}} = E^*$. This order interval is of the form $[0, \frac{1}{a_0}\mathbf{1}]$. We define the following price-adjustment correspondence $\phi : D_1 \to 2^{D_1}$, where $\phi(\pi) = \{\frac{1}{1+\|z\|}\pi | z \in z(\pi)\}$. We have to prove that the correspondence ϕ has fixed points. For this purpose, we use [8, Th.1]. Consider a net $\pi_a \overset{\sigma(E^*,E)}{\to} \pi$ for which $z_a \in z(\pi_a)$ and $z_a \overset{\sigma(E,E^*)}{\to} z$. Due to the weak lower-semicontinuity of the norm([1, Lem.6.22]), $\liminf_a \|z_a\| \geq \|z\|$. Since $L^{1+\varepsilon}, L^{1+\frac{1}{\varepsilon}}$ are Banach lattices, the last norm inequality implies the pair of inequalities

$$\frac{\pi_a(x^+)}{1+\|z_a\|} \leq \frac{\pi_a(x^+)}{1+\|z\|},$$

$$\frac{-\pi_a(x^-)}{1+\|z\|} \leq \frac{-\pi_a(x^-)}{1+\|z_a\|},$$

for a subnet of the index set $a \in A$, whose elements are re-labelled by a. Finally

$$\frac{\pi_a(x)}{1+\|z\|} \leq \frac{\pi_a(x)}{1+\|z_a\|} \leq \frac{\pi_a(x)}{1+\|z\|},$$

for any $x \in E$, which implies that ϕ has closed graph, because the $\sigma(E^*, E)$ -limit of the net $\frac{\pi_a}{1+\|z_a\|}$ is actually equal to $\frac{\pi}{1+\|z\|}$. This is equivalent to upper hemicontinuity ([1, Th.17.11]), for Hausdorff compact spaces. The assumption that any $\phi(\pi), \pi \in D_1$ is a $\sigma(E, E^*)$- closed set is satisfied, since $\phi(\pi) = \{\frac{\pi}{1+\|z\|}, z \in z(\pi)\}$. Thus, we have to deduce that if $\frac{\pi_a}{1+\|z_a\|}, z_a \in z(\pi_a)$ is a $\sigma(E^*, E)$-convergent net, then its limit has the form $\frac{\pi}{1+\|z\|}$, where $z \in z(\pi)$. z has non-empty values and also its values lie in the order-interval $[0, e]$, which is a $\sigma(E, E^*)$- compact set because $\langle E, E^* \rangle$ is a symmetric Riesz pair, [1, Th.8.60]. Hence, we obtain a $\sigma(E, E^*)$- convergent subnet of z_a to some z. Hence, by Pr. 3.9 and the upper hemicontinuity of the correspondence z, the $\sigma(E^*, E)$-limit of the net $\frac{\pi_a}{1+\|z_a\|}$ is equal to $\frac{\pi}{1+\|z\|}$. Due to the $\sigma(E^*, E)$ -compactness of $[0, \frac{1}{a_0}\mathbf{1}]$, which is valid because $\langle E^*, E \rangle$ is a symmetric Riesz pair too, [1, Cor.8.61], we also obtain a $\sigma(E^*, E)$ -convergent net π_a to some π (supposing that index sets of nets are unified). We also notice that D_1 is itself an almost convex set, while the assumption of density is also satisfied. For the almost convexity of D_1, we notice that if we consider some $\delta > 0, \delta \in \mathbf{R}$ and the corresponding open ball $B(0, \delta)$ in E^*, for any $g_1, g_2, ..., g_n \in D_1$, there exist $w_1, w_2, ...w_n \in D_1$, where $w_i = (1-\delta)g_i, i = 1, 2, ..., n$, where because of the fact that δ is adequately small, $1 - \delta > 0$. Hence, since

$$\delta max_{i=1,2,...,n} \|g_i\|_{1+\frac{1}{\varepsilon}} \leq \frac{1}{a_0}, (1-\delta) max_{i=1,2,...,n} \|g_i\|_{1+\frac{1}{\varepsilon}} \leq \frac{1}{a_0},$$

while since $0 \leq g_i \leq \frac{1}{a_0}\mathbf{1}$, and $L^{1+\frac{1}{\varepsilon}}$ is a Banach lattice, $max_{i=1,2,...,n} \|g_i\|_{1+\frac{1}{\varepsilon}} \leq \frac{1}{a_0}$. Moreover, since $0 < a_0 < 1$, $\frac{1}{a_0} > 1$, hence $0 < \delta, 1 - \delta \leq 1$. Also, for any $\delta \in (0, 1)$, the assumption $co\{w_1, w_2, ..., w_n\} \subseteq [0, \frac{1}{a_0}\mathbf{1}]$, since the last set is actually a convex set. Finally, we have to deduce the convexity of the values of ϕ in D_1. It suffices to verify it for a convex combination of two values

$$\frac{\pi_1}{1+\|z_1\|}, z_1 \in z(\pi_1), \ \frac{\pi_2}{1+\|z_2\|}, z_2 \in z(\pi_2), \pi_1 \neq t\pi_2, t > 0, t \in \mathbf{R}.$$

Since $\frac{1}{1+\|z\|} \leq 1, z \in z(\pi)$, we obtain that $\frac{\pi_i}{1+\|z_i\|} \in D_1, i = 1, 2$, which is convex, hence the convex combination $\frac{\lambda\pi_1}{1+\|z_1\|} + \frac{(1-\lambda)\pi_2}{1+\|z_2\|}, \lambda \in (0, 1)$ lies in $D_1 = [0, \frac{1}{a_0}\mathbf{1}]$. As a conclusion, all the assumptions of [8, Th.1] are satisfied, hence the fixed-points' set of ϕ is non-empty. The correspondence's ϕ fixed-points do not

include 0, since $z(\pi)$ is not defined on $\pi = 0$. The fixed-points of ϕ are equilibrium price functionals, since if $\pi \in \phi(\pi)$, then $\pi = \frac{1}{1+\|z\|}\pi, z \in z(\pi)$. Then $\|z\| = 0$, which implies $z = 0$ for some $z \in z(\pi)$, namely $0 \in z(\pi)$. This indicates the existence of a Risk Exchange Equilibirium.

Corollary 3.14 *Under Conditions (A), (B), (C),(D) any Expected Shortfall Risk Exchange Economy has a Risk Exchange Equilibirium in $L^{1+\varepsilon}$.*

4 Appendix

In this section, we mention some essential notions and results from the theory of partially ordered linear spaces which are used in the previous sections of this article.

Let E be a (normed) linear space. A set $C \subseteq E$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \in \mathbf{R}_+$ is called *wedge*. A wedge for which $C \cap (-C) = \{0\}$ is called *cone*. A pair $(E, \geq)$ where E is a linear space and $\geq$ is a binary relation on E satisfying the following properties:

(i) $x \geq x$ for any $x \in E$ (reflexive)

(ii) If $x \geq y$ and $y \geq z$ then $x \geq z$, where $x, y, z \in E$ (transitive)

(iii) If $x \geq y$ then $\lambda x \geq \lambda y$ for any $\lambda \in \mathbf{R}_+$ and $x + z \geq y + z$ for any $z \in E$, where $x, y \in E$ (compatible with the linear structure of E),

is called *partially ordered linear space*.

The binary relation $\geq$ in this case is a *partial ordering* on E. The set $P = \{x \in E | x \geq 0\}$ is called *(positive) wedge* of the partial ordering $\geq$ of E. Given a wedge C in E, the binary relation $\geq_C$ defined as follows:

$$x \geq_C y \Longleftrightarrow x - y \in C,$$

is a partial ordering on E, called *partial ordering induced by C on E*. If the partial ordering $\geq$ of the space E is *antisymmetric*, namely if $x \geq y$ and $y \geq x$ implies $x = y$, where $x, y \in E$, then P is a cone. If E is partially ordered by C, then any set of the form $[x, y] = \{r \in E | y \geq_C r \geq_C x\}$ where $x, y \in C$ is called *order-interval of E*. E' denotes the linear space of all linear functionals of E, while E^* is the norm dual of E^*, in case where E is a normed linear space.

Suppose that C is a wedge of E. A functional $f \in E'$ is called *positive functional* of C if $f(x) \geq 0$ for any $x \in C$. $f \in E'$ is a *strictly positive functional* of C if $f(x) > 0$ for any $x \in C \setminus \{0\}$. A linear functional $f \in E'$ where E is a normed linear space, is called *uniformly monotonic functional* of C if there is some real number $a > 0$ such that $f(x) \geq a\|x\|$ for any

$x \in C$. In case where a uniformly monotonic functional of C exists, C is a cone. $C^0 = \{f \in E^* | f(x) \geq 0$ for any $x \in C\}$ is the *dual wedge of* C *in* E^*. Also, by C^{00} we denote the subset $(C^0)^0$ of E^{**}. It can be easily proved that if C is a closed wedge of a reflexive space, then $C^{00} = C$. If C is a wedge of E^*, then the set $C_0 = \{x \in E | \hat{x}(f) \geq 0$ for any $f \in C\}$ is the *dual wedge of* C *in* E, where $\hat{}: E \to E^{**}$ denotes the natural embedding map from E to the second dual space E^{**} of E. Note that if for two wedges K, C of E $K \subseteq C$ holds, then $C^0 \subseteq K^0$.

The partially ordered vector space E is a *vector lattice* if for any $x, y \in E$, the supremum and the infimum of $\{x, y\}$ with respect to the partial ordering defined by P exist in E. In this case $\sup\{x, y\}$ and $\inf\{x, y\}$ are denoted by $x \vee y$, $x \wedge y$ respectively. If so, $|x| = \sup\{x, -x\}$ is the *absolute value* of x and if E is also a normed space such that $\| \, |x| \, \| = \|x\|$ for any $x \in E$, then E is called *normed lattice*.

Finally, we recall that the *usual partial ordering* of an $L^p(\Omega, \mathcal{F}, \mu)$ space, where $(\Omega, \mathcal{F}, \mu)$ is a probability space is the following: $x \geq y$ if and only if the set $\{\omega \in \Omega : x(\omega) \geq y(\omega)\}$ is a set lying in $\mathcal{F}$ of μ-probability 1.

All the previously mentioned notions and related propositions concerning partially ordered linear spaces are contained in [9].

A subset F of a convex set C in E is called *extreme set* or in other words *face* of C, if whenever $x = az + (1 - a)y \in F$, where $0 < a < 1$ and $y, z \in C$ implies $y, z \in F$. If F is a singleton, F is called *extreme point* of C.

References

[1] C.D. Aliprantis, K.C. Border, (2006) *Infinite Dimensional Analysis, A Hitchhiker's Guide*, Springer, New York.

[2] P. Artzner, F. Delbaen, J.M. Eber, D. Heath, (1999) Coherent measures of risk. *Mathematical Finance*, **9**, 203-228 (1999).

[3] Barbu, V., Precupanu, T. (1986) *Convexity and Optimization in Banach Spaces*, D. Riedel Publishing Company, Kluwer Academic Publishers Group.

[4] Delbaen, F. (2002) Coherent risk measures on general probability spaces. In *Advances in finance and stochastics: essays in honour of Dieter Sondermann*, Springer-Verlag: Berlin, New York, 1-38.

[5] Delbaen, F. Schachermayer, W. (1994) A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, **300** 463-520.

[6] Drapeau, S., Kupper, M., Papapantoleon, A. (2013) A Fourier Approach to the Computation of CV@R and Optimized Certainty Equivalents.

[7] Föllmer, H., Schied, A. (2002) Convex measures of risk and trading constraints. *Finance and Stochastics*, **6** 429-447.

[8] Himmelberg, C. (1972) Fixed points of compact multifunctions. *Journal of Mathematical Analysis and Applications*, **38** 205-207.

[9] Jameson, G. (1970) *Ordered Linear Spaces -Lecture Notes in Mathematics vol. 141*, Springer-Verlag

[10] Kaina, M., Rüschendorf (2009) Convex Risk Measures on L^p-spaces. *Mathematical Methods in Operations Research* **69** 475-495.

[11] Kountzakis, C. Polyrakis, I.A. (2006) Geometry of Cones and an Application in the theory of Pareto Effiecient Points. *Journal of Mathematical Analysis and Applications*, **320** 340-351.

[12] Polyrakis, I.A. (2008) Demand functions and reflexivity. *Journal of Mathematical Analysis and Applications*, **338** 695-704.

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