Binary Relations in the Space of Binary Relations. II.

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Abstract

This article formulates principles of extension, saturation and convergence, and shows how to implement them. In socio-economic systems, there are "reference groups", with the indicators of which the results of the research and experimentation are compared.

Keywords: principles of ordering, extension, saturation, convergence, reference group

The Extension Principle (P). Implementation Methods and Their Hierarchical Structure

Let $\mu$ be some measure and $\rho$ some metric defined on the set $\tilde{y} \times \tilde{y} = f_0(X) \times f_0(X)$. Let $\nu(z) = \nu(y_1, y_2) = (f_0(x_1), f_0(x_2))$ be a scalar continuous function on the set $\tilde{y} \times \tilde{y} = f_0(\tilde{X}) \times f_0(\tilde{X})$

$$\nu(z_1) \geq \nu(z_2) \iff (z_1, z_2) \in \mathbb{R}, \nu(z) \geq 0 \ \forall z \in \tilde{y} \times \tilde{y}.$$

We introduce the binary relations $R_{\text{CP1}}, R_{\text{CP2}}, R_{\text{CP3}}, R_{\text{CP4}}, R_{\text{CP5}}$.

CP1. $(R^1, R^2) \in R_{\text{CP1}} \iff R^1 \neq \tilde{y} \times \tilde{y} \& R^1 \supseteq R^2$;
$(R^1, R^2) \in R_{\text{CP1}} \iff R^1 \supseteq \tilde{y} \times \tilde{y} \& R^1 \supseteq R^2$;
$(R^1, R^2) \in R_{\text{CP1}} \iff R^1 \neq \tilde{y} \times \tilde{y} \& (R^1 = R^2) \vee \nu[R^1 \not\subset R^2 \& R^2 \not\subset R^1]$;
The proof of validity is similar to (AP1.1).

This axiom fails to hold true. Indeed, let

\[ R \subseteq \bar{y} \times \bar{y} \& (R^1 \cup R^2 \supset \{z_1\} \& R^2 = \{z_2\}) \]

or

\[ \exists \bar{z}, \bar{w} \subseteq \bar{y} : \mu(z_1, z_2) \geq \mu(z_1, z_2, z_3) \]

\[ \mu_2(z_1, z_2) = \sup_{z', z'' \in \bar{y}} \rho(z', z'') \]

**Theorem 1.** [1, 2] For the binary relations \( R_{CP1}, R_{CP2}, R_{CP3}, R_{CP4}, R_{CP5} \) the validity or invalidity of the axioms defining the extension principle (P) of ordering binary relations is given in the table below:

**Proof.** The proofs of validity of axioms (AO1.1) - (AO2.1) are obvious.

\[ R_{CP1} \]

Consider the binary relation \( R_{CP1} \).

**AP1.1** Let \( R^1, R^2 \in R_{CP1} \), then \( R^1 \neq \bar{y} \times \bar{y} \& \) \( R^1 \supset \{z_1\} \& \) \( R^2 = \{z_2\} \). But \( R^1 \cup R^2 \supset \bar{R} \), where \( R^1 \cup R^2 \neq \bar{y} \times \bar{y} \), and hence \( (R^1 \cup R^2, \bar{R}) \in R_{CP1} \).

**AP1.2** The proof of validity is similar to (AP1.1).

**AP2.1** Let \( R^1, R^2 \in R_{CP1} \) and \( R^3, R^4 \in R_{CP1} \), then \( R^1 \supset R^2 \) and \( R^3 \supset R^4 \). This means that \( R^1 \cup R^2 \supset R^4 \) or \( (R^1 \cup R^2, \bar{R}^3 \cup R^4) \in R_{CP1} \).

**AP2.2** The proof of validity is similar to (AP2.1).

**AP3.1** This axiom fails to hold true. Indeed, let \( R^1 \subset R^2 \subset \bar{R} \), then \( R^1 \cup R^2 = R^2 \cup \bar{R} \) or \( (R^1 \cup R^2, \bar{R} \cup R^3) \in J_{CP1} \), but \( R^2 \supset R^1 \), that is, \( (R^2, R^1) \in P_{CP1} \).
(AP3.2) This axiom fails to hold true. Indeed, let \( R^1 = \{z_1, z_2\}, \)
\( R^2 = R^3 = \{z_3\}, \) then \( R^1 \cup R^3 = \{z_1, z_2, z_3\} \) and \( R^2 \cup R^3 = \{z_3\}, \) that is, \( R^1 \cup R^3 \supset R^2 \cup R^3 \) or \( (R^1 \cup R^3, R^2 \cup R^3) \in P_{CP1}, \) but \( R^1 \cap R^2 = \emptyset, \) and hence \( (R^1, R^2) \in J_{CP1}. \)

(AP4.1) Let \( (R^1, R^2) \in R_{CP1}, \) then \( R^1 \not= \bar{y} \times \bar{y} \) and \( R^1 \supseteq R^2, \) hence \( R^3 \setminus R^1 \subseteq R^3 \setminus R^2, \) that is, \( (R^3 \setminus R^2, R^3 \setminus R^1) \in R_{CP1}. \)

(1) The proof of validity is similar to (AP4.1).

(2) Consider the binary relation \( R_{CP2}. \)

(3) Let \( (R^1, R^3) \in R_{CP2}, \) then \( R^1 \supseteq \{z_1\} \) and \( R^3 = \{z_3\}. \) Let \( R^2 \supseteq \{z_2\}, \) then \( R^1 \cup R^2 \supseteq \{z_1\} \cup \{z_2\} \supseteq \{z_1\}, \) hence \( (R^1 \cup R^2, R^3) \in P_{CP2}. \)

(3.1) From the proof of the previous case it will be seen that for \( R_{CP2} \)
this axiom is the weakening of axiom (AP1.1) and, therefore, is also valid.

(2.1) This axiom fails to hold true. Indeed, let \( R^1 = R^3 = \{z_1\}, \)
\( R^2 = \{z_2\} \) and \( R^4 = \{z_4\}, \) then \( R^1 \cup R^3 = \{z_1\} \), \( R^2 \cup R^4 = \{z_2, z_4\} \) and \((R^2 \cup R^4, R^1 \cup R^3) \in R_{CP2}.\)

(2.2) This axiom fails to hold true. Indeed, let \( R^1 = \{z_1, z_3\}, \)
\( R^2 = \{z_2\}, \) \( R^3 = \{z_1, z_3\} \) and \( R^4 = \{z_4\}, \) then \( R^1 \cup R^3 \supset \{z_1\} \) and \( R^2 \cup R^4 \supset \{z_2\}, \) that is, \((R^1 \cup R^3, R^2 \cup R^4) \in J_{CP2}.\)

(3.1) Let, \( (R^1 \cup R^3, R^2 \cup R^3) \in R_{CP2}, \) then \( R^1 \cup R^3 \supseteq \{z_1\} \) and \( R^2 \cup R^3 = \{z_3\}. \) But from the latter equality it, follows that, \( R^2 = R^3 = \{z_2\}, \) or one of the relations is empty. For \( R^1, \) however, we have \( R^1 \supseteq \{z_1\}, \) that is, in any case \( (R^1, R^2) \in R_{CP2}.\)

(3.2) This axiom fails to hold true. Indeed, as can be seen from the
proof of the previous case, if \( R^1 = \{z_1\}, \) then \( (R^1, R^2) \in J_{CP2}.\)

(4.1) This axiom fails to hold true. Indeed, let \( R^1 = \{z_1, z_3\}, \)
\( R^2 = \{z_2\}, \) then \( (R^1, R^2) \in P_{CP2}. \) Let \( R^3 = \{z_1, z_4\}, \) then \( R^3 \setminus R^1 = \{z_2, z_4\}, \)
but \( R^3 \setminus R^2 = \{z_1\}, \) hence \( (R^3 \setminus R^1, R^3 \setminus R^2) \in P_{CP2}. \)

(4.2) The proof of invalidity is similar to (AP4.1).

(3.3) Consider the binary relation \( R_{CP3}. \)

(1) Let \( (R^1, R^3) \in R_{CP3}, \) then \( \mu(R^1) \geq \mu(R^3). \) But \( \mu(R^1 \cup R^2) \geq \mu(R^1), \) hence \( \mu(R^1 \cup R^2) \geq \mu(R^3), \) that is, \((R^1 \cup R^2, R^3) \in R_{CP3}.\)

(1.1) The proof of validity is similar to (AP1.1).

(2.1) This axiom fails to hold true. Indeed, let us consider the
set of finite binary relations. Then, instead of a measure, we may consider their
power. Let \( R^1 = R^3 = \{z_1, z_2, z_3\}, \) \( R^2 = \{z_4, z_5\} \) and \( R^4 = \{z_5, z_6\}, \) then \( \mu(R^1) = 3 \geq 2 = \mu(R^2) \) and \( \mu(R^3) = 3 \geq 2 = \mu(R^4). \) But \( \mu(R^1 \cup R^3) = 3 < 4 = \mu(R^2 \cup R^4). \)

(2.2) The proof of invalidity is similar to (AP1.1).

(3.1) This axiom fails to hold true. Indeed, let us again consider the
set of finite binary relations. Let \( R^1 = \{z_1\} \) and \( R^2 = R^3 = \{z_2, z_3\}, \) then \( \mu(R^1 \cup R^3) = 3 \geq 2 = \mu(R^2 \cup R^3). \) But \( \mu(R^1) = 1 < 2 = \mu(R^2). \)

(3.2) The proof of invalidity is similar to (AP3.1).
such that $R \subseteq R^3$. Suppose to the contrary that the axiom holds for the binary relations we have chosen, that is, $\mu(R^1) \geq \mu(R^2)$ implies $\mu(R^3 \setminus R^1) \leq \mu(R^3 \setminus R^2)$. But $\mu(R^3 \setminus R^1) = \mu(R^3) = \mu(R^3 \setminus R^2) + \mu(R^3) > \mu(R^3 \setminus R^2)$.

\textbf{(AP4.2)} The proof of invalidity is similar to (AP4.1).

(\mathcal{R}_{CP4}) Consider the binary relation $\mathcal{R}_{CP4}$.

\textbf{(AP1.1)} Let $(R^1, R^2) \subseteq \mathcal{R}_{CP4}$, then $\int_{R^1} \nu(z)dz \geq \int_{R^2} \nu(z)dz$. But $\nu(z) \geq 0$, therefore $\int_{R^1 \cup R^2} \nu(z)dz \geq \int_{R^3} \nu(z)dz$, and hence $\int_{R^1 \cup R^2} \nu(z)dz \geq \int_{R^3} \nu(z)dz$, that is, $(R^1 \cup R^2, R^3) \subseteq \mathcal{R}_{CP4}$.

\textbf{(AP1.2)} The proof of validity is similar to (AP1.1).

\textbf{(AP2.1)}, \textbf{(AP2.2)}, \textbf{(AP3.1)}, \textbf{(AP3.2)}, \textbf{(AP4.1)}, \textbf{(AP4.2)} These axioms fail to hold true because with $\nu(z) = 1$ for $\forall z \in \tilde{y} \times \tilde{y}$ we have $u_{CP3}(R) = u_{CP4}(R)$ for all $R$, and they fail to hold for $\mathcal{R}_{CP3}$.

(\mathcal{R}_{CP5}) Consider the binary relation $\mathcal{R}_{CP5}$.

\textbf{(AP1.1)} Let $(R^1, R^2) \subseteq \mathcal{R}_{CP5}$, then $\sup_{R^1 \cup R^2} \rho(z', z'') \geq \sup_{R^3} \rho(z', z'')$.

But $\sup_{z', z'' \in R^1 \cup R^2} \rho(z', z'') \geq \sup_{z', z'' \in R^1} \rho(z', z'')$, that is, $(R^1 \cup R^2, R^3) \subseteq \mathcal{R}_{CP5}$.

\textbf{(AP2.1)} The proof of validity is similar to (AP1.1).

\textbf{(AP3.1)} This axiom fails to hold true. Indeed, let us choose binary relations such that $R^1 = R^3$, $R^2 \cap R^4 = \emptyset$, in which case $\sup_{z', z'' \in R^1} \rho(z', z'') = 3$, and $\sup_{z', z'' \in R^3} \rho(z', z'') = 2$.

Then $(R^1, R^2) \subseteq \mathcal{R}_{CP5}$, $(R^3, R^4) \subseteq \mathcal{R}_{CP5}$, but $(R^2 \cup R^1, R^3 \cup R^4) \notin \mathcal{R}_{CP5}$, because

$$\sup_{z', z'' \in R^3 \cup R^4} \rho(z', z'') = 3 < 4 = \sup_{z', z'' \in R^3 \cup R^4} \rho(z', z'') + \sup_{z', z'' \in R^3} \rho(z', z'').$$

\textbf{(AP2.2)} The proof of invalidity is similar to (AP2.1).

\textbf{(AP3.1)} This axiom fails to hold true. Indeed, let us choose binary relations such that $R^1 = \{z_1\} \notin R^3$, $R^2 = R^3$, in which case $\sup_{z \in R^3} \rho(z_1, z) = 2$, and $\sup_{z', z'' \in R^3} \rho(z', z'') = 1$. Then $(R^1 \cup R^3, R^2 \cup R^3) \notin \mathcal{R}_{CP5}$, because

$$\sup_{z', z'' \in R^3 \cup R^4} \rho(z', z'') = \sup_{z_1 \in R^3} \rho(z_1, z) = 2 > 1 = \sup_{z', z'' \in R^3 \cup R^4} \rho(z', z'').$$

But $(R^2, R^1) \in \mathcal{P}_{CP5}$, because $\sup_{z', z'' \in R^1} \rho(z', z'') = 0 < 1 = \sup_{z', z'' \in R^2} \rho(z', z'')$. 
(AP3.2) The proof of invalidity is similar to (AP3.1).

(AP4.1) This axiom fails to hold true. Indeed, let \( R^1 \cap R^3 = \emptyset \) and \( R^2 \cap R^3 \neq \emptyset \), then \( R^3 = (R^3 \setminus R^1) \supset (R^3 \setminus R^2) \), hence
\[
\sup_{z', z'' \in R^3 \setminus R^1} \rho(z', z'') \geq \sup_{z', z'' \in R^3 \setminus R^2} \rho(z', z''),
\]
in which case we may always choose binary relations \( R^1, R^2 \) and \( R^3 \) to be such that the latter inequality is satisfied as the strict inequality.

(AP4.2) The proof of invalidity is similar to (AP4.1).

This completes the proof of the theorem.

Thus, as in the case for the matching principle, none of the above binary relations \( R_{CPj}, j = \bar{1,5} \), each of which is conceptually intended to assess the "width" of comparable sets, satisfies the set of axioms by which the extension principle was defined. In order to weaken the initially formulated axioms, we shall identify a number of conditions that have led to invalidity of some axioms in the collection (AP1.1) - (AP4.2):
- indifference of some axioms in the sense of \( R_{CPj}, j = \bar{1,5} \);
- absence of requirements for intersection and/or union of comparable binary relations.

In what follows, we shall construct the weakened collections of axioms by taking into account the above specific features. First, consider the binary relation \( R_{CP1} \) and weaken the invalidity of axiom (AP3.1), (AP3.2) for it in the following way.

(\textbf{AP3.1} (1)). If \( (R^1 \cup R^3, R^2 \cup R^3) \in \mathcal{P}, R^1 \neq \emptyset, R^1 \neq R^2, \) then \( (R^1, R^2) \in \mathcal{R} \).

(\textbf{AP3.1} (2)). If \( (R^1 \cup R^3, R^2 \cup R^3) \in \mathcal{P}, (R^3 \setminus R^1) \cap R^2, R^1 \neq \emptyset, R^1 \neq R^2, \) then \( (R^1, R^2) \in \mathcal{P} \).

We now consider the binary relation \( R_{C2P} \). As can be seen from the proof of Theorem 1, in order to satisfy axiom (AP2.1) it suffices to let \( R^2 \cup R^1 = \{z_2\} \) and \( R^1 \cap R^3 \supset \{z_1\} \). But under the preconditions of axiom \( (R^1, R^2) \in \mathcal{R} \) and \( (R^3, R^4) \in \mathcal{R} \) the second requirement is always satisfied. The first requirement, however, means that we must have \( R^2 = R^4 = \{z_2\} \). A similar situation also holds for axiom (AP2.2). To satisfy axiom (AP3.2), let \( R^1 \supset \{z_1\} \), and to satisfy axiom (AP4.1), let \( R^3 \setminus R^1 = \{z_3\} \) and \( R^1 = \{z_1\} \). Note that under the precondition \( (R^1, R^2) \in \mathcal{P} \) the second requirement for axiom (AP4.1) is always satisfied. The first requirement, however, means that we must have \( R^3 = R^1 \cap R^3 \cup \{z\} \). This relationship may be insufficient for axiom (AP4.2). Indeed, let \( R^1 = \{z_1, z_2\}, R^2 = \{z_3\} \) and \( R^3 = \{z_2, z_3\} \), then \( (R^1, R^2) \in \mathcal{P}_{CP2}, R^2 \setminus R^1 \neq \emptyset, R^3 = R^1 \cap R^3 \cup \{z\} \), but \( (R^3, R^2, R^3 \setminus R^1) \in \mathcal{J}_{CP2} \). Therefore, we incorporate into axiom (AP4.2) another requirement: \( R^3 \supset R^2 \cap \{z'\} \) eliminating as a result the condition \( R^3 \setminus R^1 \neq R^3 \setminus R^2 \).

(\textbf{AP2.1} (2)). If \( (R^1, R^2) \in \mathcal{R} \), \( (R^3, R^2) \in \mathcal{R} \), \( R^1 \cup R^3 \neq \bar{y} \times \bar{y} \), then \( (R^1 \cup R^3, R^2) \in \mathcal{R} \).
\textbf{AP2.2(2).} If \((R^1, R^2) \in \mathcal{P}, (R^3, R^2) \in \mathcal{R}, R^1 \cup R^3 \neq \tilde{y} \times \tilde{y}, R^1 \cup R^3 \neq R^2 \cup R^4\), then \((R^1 \cup R^3, R^2 \cup R^4) \in \mathcal{P}\).

\textbf{AP3.2(2).} If \((R^1 \cup R^2, R^2 \cup R^3) \in \mathcal{P}, R^1 \ni \{z_1\}, R^1 \neq \emptyset, R^1 \neq R^2, \text{ then } \(R^1, R^2) \in \mathcal{P}\).

\textbf{AP4.1(2).} If \((R^1, R^2) \in \mathcal{R}, R^3 = R^1 \cap R^3 \cup \{z\}, R^3 \setminus R^2 \neq \emptyset, \text{ then } \(R^3 \setminus R^2, R^3 \setminus R^1) \in \mathcal{R}\).

\textbf{AP4.2(2).} If \((R^1, R^2) \in \mathcal{P}, R^3 = R^1 \cap R^3 \cup \{z\}, R^3 \ni R^2 \cap R^3 \cup \{z'\}, R^3 \setminus R^2 \neq \emptyset, \text{ then } \(R^3 \setminus R^2, R^3 \setminus R^1) \in \mathcal{P}\).

We will consider further the system of weakened axioms as follows.

\textbf{AP2.1(3).} If \((R^1, R^2) \in \mathcal{R}, (R^3, R^2) \in \mathcal{R}, \mu(R^1 \cap R^3) = \mu(R^2 \cap R^4), R^1 \cup R^3 \neq \tilde{y} \times \tilde{y}, \text{ then } \(R^1 \cup R^3, R^2 \cup R^4) \in \mathcal{R}\).

\textbf{AP2.2(3).} If \((R^1, R^2) \in \mathcal{P}, (R^3, R^4) \in \mathcal{R}, \mu(R^1 \cap R^3) = \mu(R^2 \cap R^4), R^1 \cup R^3 \neq \tilde{y} \times \tilde{y}, R^1 \cup R^3 \neq R^2 \cup R^4, \text{ then } \(R^1 \cup R^3, R^2 \cup R^4) \in \mathcal{P}\).

\textbf{AP3.1(3).} If \((R^1 \cap R^3, R^2 \setminus R^3) \in \mathcal{R}, \mu(R^1 \cap R^3) = \mu(R^2 \cap R^3), \text{ then } \(R^1, R^2) \in \mathcal{R}\).

\textbf{AP3.2(3).} If \((R^1 \cup R^3, R^2 \cup R^3) \in \mathcal{P}, \mu(R^1 \cap R^3) = \mu(R^2 \cap R^3), R^1 \neq R^2, \text{ then } \(R^1, R^2) \in \mathcal{P}\).

\textbf{AP4.1(3).} If \((R^1, R^2) \in \mathcal{R}, \mu(R^1 \cup R^3) \leq \mu(R^2 \cup R^3), R^3 \setminus R^2 \neq \emptyset, \text{ then } \(R^3 \setminus R^2, R^3 \setminus R^1 \in \mathcal{R}\).

\textbf{AP4.2(3).} If \((R^1, R^2) \in \mathcal{P}, \mu(R^1 \cup R^3) \leq \mu(R^2 \cup R^3), R^3 \setminus R^2 \neq \emptyset, R^3 \setminus R^1 \neq R^3 \setminus R^2, \text{ then } \(R^3 \setminus R^2, R^3 \setminus R^1) \in \mathcal{P}\).

Since \(\nu(z) > 0\) for \(\forall z \in \tilde{y} \times \tilde{y}\), assumptions could be made by analogy with axioms (AP2.1(3) - AP4.2(3)) as the system of weakened axioms defining the extension principle and holding for the binary relation \(\mathcal{R}_{CP3}\). This would lead to specification in the context of axioms of the requirements for equality of integrals in binary relations. But the resulting axioms would regulate not only particular properties of binary relations, but also those of functions \(\nu(z)\), which, in turn, would significantly restrict their choice. For this reason, we shall incorporate into the appropriate axioms some other conditions, such as \(\mu(R^1 \cap R^2) = 0\). We shall add to axioms (AP4.1) and (AP4.2) a special case of requirement embedded in (AP4.1(3)) and (AP4.2(3)): \(R^1 \subseteq R^3\).

\textbf{AP2.1(4).} If \((R^1, R^2) \in \mathcal{R}, (R^3, R^4) \in \mathcal{R}, \mu(R^1 \cap R^3) = 0, \mu(R^2 \cap R^4) = 0, R^1 \cup R^3 \neq \tilde{y} \times \tilde{y}, \text{ then } \(R^1 \cup R^3, R^2 \cup R^4) \in \mathcal{R}\).

\textbf{AP2.2(4).} If \((R^1, R^2) \in \mathcal{P}, (R^3, R^4) \in \mathcal{R}, \mu(R^1 \cap R^3) = 0, \mu(R^2 \cap R^4) = 0, R^1 \cup R^3 \neq \tilde{y} \times \tilde{y}, R^1 \cup R^3 \neq R^2 \cup R^4, \text{ then } \(R^1 \cup R^3, R^2 \cup R^4) \in \mathcal{P}\).

\textbf{AP3.1(4).} If \((R^1 \cup R^3, R^2 \cup R^3) \in \mathcal{R}, \mu(R^1 \cap R^3) = \mu(R^2 \cap R^3) = 0, \text{ then } \(R^1, R^2) \in \mathcal{R}\).

\textbf{AP3.2(4).} If \((R^1 \cup R^3, R^2 \cup R^3) \in \mathcal{P}, \mu(R^1 \cap R^3) = \mu(R^2 \cap R^3) = 0, R^1 \neq R^2, \text{ then } \(R^1, R^2) \in \mathcal{P}\).

\textbf{AP4.1(4).} If \((R^1, R^2) \in \mathcal{R}, R^1 \subseteq R^3, R^3 \setminus R^2 \neq \emptyset, \text{ then } \(R^3 \setminus R^2, R^3 \setminus R^1) \in \mathcal{R}\).
**AP4.2** If \((R^1, R^2) \in \mathcal{P}, R^1 \subseteq R^2, R^3 \cap R^2 \neq \emptyset, R^3 \cap R^1 \neq R^3 \cap R^2\), then \((R^3 \cap R^2, R^3 \cap R^1) \in \mathcal{P}.

We will now consider the binary relation \(\mathcal{R}_{CP_5}\). As can be seen from the proof of Theorem 1, and by analogy with the previous cases, we need to incorporate into axioms primarily conditions such that:

\[
\sup_{z', z'' \in R^1 \cup R^2} \rho(z', z'') = \sup_{z', z'' \in R^1} \rho(z', z'') + \sup_{z', z'' \in R^2} \rho(z', z'').
\]

To this end, let us denote:

\[
\rho(z', z'') = \sup_{z', z'' \in R^1} \rho(z', z''), \quad \text{where } \{z_1', z_1''\} \subseteq \overline{R^1} \text{ and } \{z_2', z_2''\} \subseteq \overline{R^2}.
\]

Also, let \([z_1', z_1''], [z_2', z_2''], [z_1', z_2']\) and \([z_1', z_2'']\) be the segments connecting the pairs of points \((z_1', z_1''), (z_2', z_2''), (z_1', z_2')\) and \((z_1', z_2'')\) respectively, regardless of their relative positions, and let \(\rho[z_1', z_1''], \rho[z_2', z_2''], \rho[z_1', z_2']\) and \(\rho[z_1', z_2'']\) be the lengths of these segments, respectively.

Let \(z_0 \in [z_1', z_2']\). Suppose also that \(z_0 \in [z_1', z_2'']\). The later assumption means that the generating diameters of binary relations \(R^1\) and \(R^2\) lie on the same straight line. If, however, we now further assume that

\((R^1 \cap R^2) \cap [z_1', z_2'' \cap \{z_0\}, \text{ then the desired relationship will hold for the binary relations } R^1 \text{ and } R^2\).

In what follows, if \((R^1 \cap R^2) \cap [z_1', z_2'' \cap \{z_0\}, \text{ then, to simplify notation, we shall write: } \rho(R^1 \cap R^2) = \{z_0\}, \text{ and when } (R^1 \cap R^2) \cap [z_1', z_2'' \cap \{z_0\}, \text{ we write } \rho(R^1 \cap R^2) = \{z_0\}.

**AP2.1** If \((R^1, R^2) \in \mathcal{R}, (R^3, R^4) \in \mathcal{R}, \rho(R^1 \cap R^3) = \{z_{13}\}, \rho(R^2 \cap R^4) = \{z_{24}\}, R^1 \cup R^3 \neq \bar{y} \times \bar{y}, \text{ then } (R^1 \cup R^3, R^2 \cup R^4) \in \mathcal{R}.

**AP2.2** If \((R^1, R^2) \in \mathcal{P}, (R^3, R^4) \in \mathcal{R}, \rho(R^1 \cap R^3) = \{z_{13}\}, \rho(R^2 \cap R^4) = \{z_{24}\}, R^1 \cup R^3 \neq R^2 \cup R^4, \text{ then } (R^1 \cup R^3, R^2 \cup R^4) \in \mathcal{P}.

**AP3.1** If \((R^1 \cup R^3, R^2 \cup R^3) \in \mathcal{R}, \rho(R^1 \cap R^3) = \{z_{13}\}, \rho(R^2 \cap R^3) = \{z_{23}\}, \text{ then } (R^1, R^2) \in \mathcal{R}.

**AP3.2** If \((R^1 \cup R^3, R^2 \cup R^3) \in \mathcal{P}, \rho(R^1 \cap R^3) = \{z_{13}\}, \rho(R^2 \cap R^3) = \{z_{23}\}, R^1 \neq R^2, \text{ then } (R^1, R^2) \in \mathcal{P}.

**AP4.1** If \((R^1, R^2) \in \mathcal{R}, \rho(R^1 \cap R^2) = \{z_{13}\}, \rho(R^2 \cap R^3) = \{z_{23}\}, R^3 \cap R^2 \neq \emptyset, \text{ then } (R^3 \cap R^2, R^3 \cap R^1) \in \mathcal{R}.

**AP4.2** If \((R^1, R^2) \in \mathcal{P}, \rho(R^1 \cap R^2) = \{z_{13}\}, \rho(R^2 \cap R^3) = \{z_{23}\}, R^3 \cap R^2 \neq \emptyset, \text{ then } (R^3 \cap R^2, R^3 \cap R^1) \in \mathcal{P}.

**Theorem 2** (Implementation of the extension principle). [1, 2] The binary relations \(\mathcal{R}_{CP_1}, \mathcal{R}_{CP_2}, \mathcal{R}_{CP_3}, \mathcal{R}_{CP_4}\) and \(\mathcal{R}_{CP_5}\) are the implementation methods for the extension principle (P) of ordering binary relations under the respective systems of axioms:

- \(\{AP(1)\} = \{AP1.1, AP1.2, AP2.1, AP2.2, AP3.1, AP3.2(1), AP4.1, AP4.2\};
- \(\{AP(2)\} = \{AP1.1, AP1.2, AP2.1(2), AP2.2(2), AP3.1, AP3.2(2), AP4.1(2), AP4.2(2)\}.

\[ \{ AP^{(j)} \} = \{ AP1.1, AP1.2, AP2.1^{(j)}, AP2.2^{(j)}, AP3.1^{(j)}, AP3.2^{(j)}, AP4.1^{(j)}, AP4.2^{(j)} \}, \forall j = 3, 5. \]

**Proof.** It can be readily seen that each of the axioms introduced before the theorem is the weakening of an appropriate axiom in the collection defining the extension principle. Therefore, to prove the theorem, it suffices to show that the foregoing weakened counterparts of axioms defining the extension principle hold for the binary relations \( R_{CP1}, R_{CP2}, R_{CP3}, R_{CP4}, R_{CP5} \).

**AP3.2**(1). Let \( (R^1 \cup R^3, R^2 \cup R^3) \in P_{CP1} \), then \( R^1 \cup R^3 \supseteq R^2 \cup R^3 \). If \( z \in R^2 \), then \( z \in R^2 \cup R^3 \supseteq R^1 \cup R^3 \). But \( (R^3 \setminus R^1) \cap R^2 = \emptyset \), therefore \( z \in R^2 \setminus R^1 \), and hence \( z \in R^1 \), and by arbitrariness \( z : R^1 \subseteq R^1 \), in which case the inclusion \( R^1 \cup R^3 \supseteq R^2 \cup R^3 \) ensures the rigor of binary relation.

**AP2.1**(2). Let \( (R^1, R^2) \in R_{CP2} \) and \( (R^3, R^2) \in R_{CP2} \), then \( R^1 \subseteq \{ z_1 \} \) and \( R^2 \subseteq \{ z_2 \} \). But then \( R^1 \cup R^3 \supseteq \{ z_1 \} \) and \( R^2 \cup R^3 \supseteq \{ z_2 \} \). It then follows from the latter equality that \( R^2 = R^3 = \{ z_2 \} \) or one of the relations is empty. For \( R^1 \), however, \( R^1 \subseteq \{ z_1 \} \), that is, in any case \( (R^1, R^2) \in P_{CP2} \).

**AP3.2**(2). Let \( (R^1 \cup R^3, R^2 \cup R^3) \in P_{CP2} \), then \( R^1 \cup R^3 \supseteq \{ z_1 \} \) and \( R^2 \cup R^3 = \{ z_2 \} \). But it follows from the latter equality that \( R^2 = R^3 = \{ z_2 \} \) or one of the relations is empty. For \( R^1 \), however, \( R^1 \subseteq \{ z_1 \} \), that is, in any case \( (R^1, R^2) \in P_{CP2} \).

**AP4.1**(2). Let \( (R^1, R^2) \in R_{CP2} \), then \( R^1 \subseteq \{ z_1 \} \) and \( R^2 = \{ z_2 \} \). Also, let \( R^3 = R^1 \cap R^3 \cup \{ z \} \), and hence \( R^3 \setminus R^1 = \{ z \} \) and \( (R^3 \setminus R^2, R^3 \setminus R^1) \in R_{CP2} \).

**AP4.2**(2). The proof of validity is similar to (AP4.1(2)), in which case the condition \( R^3 \supseteq R^2 \cap R^3 \) in \( \{ z' \} \) ensures the rigour of binary relation.

**AP2.1**(3). Let \( (R^1, R^2) \in R_{CP3} \) and \( (R^3, R^4) \in R_{CP3} \), then \( \mu(R^1) \geq \mu(R^2) \) and \( \mu(R^3) \geq \mu(R^4) \). But \( \mu(R^1 \cap R^3) = \mu(R^2 \cap R^4) \), and hence

\[
\mu(R^1 \cup R^3) = \mu(R^1) + \mu(R^3) - \mu(R^1 \cap R^3) \geq \mu(R^2) + \mu(R^4) - \mu(R^2 \cap R^4) = \mu(R^2 \cap R^4),
\]

that is, \( (R^1 \cup R^3, R^2 \cup R^4) \in R_{CP3} \).

**AP2.2**(3). The proof of validity is similar to (AP2.1(3)).

**AP3.1**(3). Let \( (R^1 \cap R^3, R^2 \cap R^3) \in R_{CP3} \), then \( \mu(R^1 \cap R^3) \geq \mu(R^2 \cap R^4) \). But \( \mu(R^1 \cap R^3) = \mu(R^2 \cap R^3) \), and hence

\[
\mu(R^1) + \mu(R^3) - \mu(R^1 \cap R^3) = \mu(R^1 \cup R^3) \geq \mu(R^2 \cup R^3) = \mu(R^2) + \mu(R^3) - \mu(R^2 \cap R^3)
\]

or \( \mu(R^1) \geq \mu(R^2) \), that is, \( (R^1, R^2) \in R_{CP3} \).

**AP3.2**(3). The proof of validity is similar to (AP3.1(3)).

**AP4.1**(3). Let \( (R^1, R^2) \in R_{CP3} \), then \( \mu(R^1) \geq \mu(R^2) \), and hence

\[
\mu(R^1 \setminus R^3) = \mu(R^2 \cup R^3) \geq \mu(R^1 \cup R^3) = \mu(R^3 \setminus R^1),
\]
that is, $(R^3 \setminus R^2, R^3 \setminus R^1) \in \mathcal{R}_{CP3}$.

**AP4.2**. The proof of validity is similar to (AP4.1).

**AP2.1**. Let $(R^1, R^2) \in \mathcal{R}_{CP4}$ and $(R^3, R^4) \in \mathcal{R}_{CP4}$, then
\[
\int_{R^1} \nu(z) dz \geq \int_{R^3} \nu(z) dz \text{ and } \int_{R^2} \nu(z) dz \geq \int_{R^4} \nu(z) dz.
\]
But
\[
\mu(R^1 \cap R^3) = \mu(R^2 \cap R^4) = 0, \text{ and hence}
\]
\[
\int_{R^1 \cup R^3} \nu(z) dz = \int_{R^3} \nu(z) dz + \int_{R^1} \nu(z) dz \geq \int_{R^1 \cup R^3} \nu(z) dz = \int_{R^3} \nu(z) dz,
\]
that is, $(R^1 \cup R^3, R^2 \cup R^4) \in \mathcal{R}_{CP4}$.

**AP2.2**. The proof of validity is similar to (AP2.1).

**AP3.1**. Let $(R^1 \cup R^3, R^2 \cup R^3) \in \mathcal{R}_{CP4}$, then
\[
\int_{R^1 \cup R^3} \nu(z) dz \geq \int_{R^3} \nu(z) dz.
\]
But $\mu(R^1 \cup R^3) = \mu(R^2 \cap R^3) = 0$, and hence
\[
\int_{R^1 \cup R^3} \nu(z) dz = \int_{R^3} \nu(z) dz + \int_{R^1} \nu(z) dz \geq \int_{R^2 \cup R^3} \nu(z) dz = \int_{R^3} \nu(z) dz,
\]
or
\[
\int_{R^1} \nu(z) dz \geq \int_{R^3} \nu(z) dz, \text{ that is, } (R^1, R^3) \in \mathcal{R}_{CP4}.
\]

**AP3.2**. The proof of validity is similar to (AP3.1).

**AP4.1**. Let $(R^1, R^2) \in \mathcal{R}_{CP4}$, then $\mu(R^1) \geq \mu(R^2)$, $R^1 \subseteq R^3$, therefore
\[
\int_{R^1 \setminus R^3} \nu(z) dz = \int_{R^1 \cup R^3} \nu(z) dz - \int_{R^3} \nu(z) dz \leq \int_{R^1 \setminus R^3} \nu(z) dz - \int_{R^1} \nu(z) dz = \int_{R^3 \setminus R^1} \nu(z) dz,
\]
that is, $(R^3 \setminus R^2, R^3 \setminus R^1) \in \mathcal{R}_{CP4}$.

**AP4.2**. The proof of validity is similar to (AP4.1).

**AP2.1**. Let $(R^1, R^2) \in \mathcal{R}_{CP5}$ and $\mathcal{R}_{CP5}$, then
\[
\sup_{z', z'' \in R^1} \rho(z', z'') \geq \sup_{z', z'' \in R^2} \rho(z', z'') \text{ and}
\]
\[
\sup_{z', z'' \in R^3} \rho(z', z'') \geq \sup_{z', z'' \in R^4} \rho(z', z'').
\]
But $\rho(R^1 \cap R^3) = \{z_{13}\}$ and $\rho(R^2 \cap R^4) = \{z_{24}\}$, therefore
\[
\sup_{z', z'' \in R^1 \cup R^3} \rho(z', z'') \geq \sup_{z', z'' \in R^1} \rho(z', z'') + \sup_{z', z'' \in R^3} \rho(z', z'') \geq \sup_{z', z'' \in R^2} \rho(z', z'') + \sup_{z', z'' \in R^4} \rho(z', z''),
\]
that is, $(R^1 \cup R^3, R^2 \cup R^4) \in \mathcal{R}_{CP5}$.

**AP2.2**. The proof of validity to (AP2.1).
\textbf{AP3.1}^{(5)}$. Let \((R^1 \cup R^3, R^2 \cup R^3) \in \mathcal{R}_{CP5}\), then

\[
\sup_{z',z'' \in R^1 \cup R^3} \rho(z', z'') \geq \sup_{z',z'' \in R^2 \cup R^3} \rho(z', z'').
\]

But \(\rho(R^1 \cap R^3) = \{z_{13}\}\) and \(\rho(R^2 \cap R^3) = \{z_{23}\}\), therefore

\[
\sup_{z',z'' \in R^1} \rho(z', z'') + \sup_{z',z'' \in R^3} \rho(z', z'') = \sup_{z',z'' \in R^1 \cup R^3} \rho(z', z'') \geq \sup_{z',z'' \in R^2} \rho(z', z'') = \sup_{z',z'' \in R^2 \cup R^3} \rho(z', z''),
\]
or

\[
\sup_{z',z'' \in R^1} \rho(z', z'') \geq \sup_{z',z'' \in R^2} \rho(z', z''), \text{ that is, } (R^1, R^2) \in \mathcal{R}_{CP5}.
\]

\textbf{AP3.2}^{(5)}. The proof of validity is similar to (AP3.1^{(5)}).

\textbf{AP4.1}^{(5)}. Let \(\rho(R^1 \cap R^3) = \{z_{13}\}\) and \(\rho(R^2 \cap R^3) = \{z_{23}\}\), therefore

\[
\sup_{z',z'' \in R^3 \setminus R^1} \rho(z', z'') = \sup_{z',z'' \in R^3 \cup R^1} \rho(z', z'') - \sup_{z',z'' \in R^1} \rho(z', z'') = \rho[z_{13}, z_{23}] - \sup_{z',z'' \in R^1} \rho(z', z'').
\]

Similarly, we may have that

\[
\sup_{z',z'' \in R^3 \setminus R^2} \rho(z', z'') = \sup_{z',z'' \in R^3 \cup R^2} \rho(z', z'') - \sup_{z',z'' \in R^2} \rho(z', z'') = \rho[z_{23}, z_{33}] - \sup_{z',z'' \in R^2} \rho(z', z'').
\]

But since \((R^1, R^2) \in \mathcal{R}_{CP5}\), that is, \(\sup_{z',z'' \in R^1} \rho(z', z'') \geq \sup_{z',z'' \in R^2} \rho(z', z'')\), and \(\rho[z_{13}, z_{23}] \leq \rho[z_{23}, z_{33}]\), we may get by combing the two relationships

\[
\sup_{z',z'' \in R^3 \setminus R^1} \rho(z', z'') \geq \sup_{z',z'' \in R^3 \setminus R^2} \rho(z', z''), \text{ that is, } (R^3 \setminus R^1, R^3 \setminus R^2) \in \mathcal{R}_{CP5}.
\]

\textbf{AP4.2}^{(5)}. The proof of validity is similar to (AP4.1^{(5)}).

This completes the proof of the theorem.

We shall construct the hierarchical structure for the foregoing implementational methods of the extension principle. The strongest method is the method (CP1) because \(q\{AP^{(1)}\} = 2\). The next level is the method (CP2): \(q\{AP^{(2)}\} = 5\). In order to construct further levels, note that the methods (CP3) - (CP5) implement the extension principle when the six axioms (AP2.1) - (AP4.2) are weakened. It can be readily seen that the system of axioms \(\{AP^{(5)}\}\) is the weakening of \(\{AP^{(4)}\}\) which, in turn, is the weakening of \(\{AP^{(3)}\}\). Therefore, the strongest of the methods is the method (CP3), then comes the method (CP4), and finally (CP5). Thus, the hierarchical structure for the implementational methods of the extension principle can be represented as follows:

\[(CP1) \mapsto (CP2) \mapsto (CP3) \mapsto (CP4) \mapsto (CP5).\]
References

[1] V.V. Kolbin, A.V. Shagov, Decision Models, Saint-Petersburg State University (2002), 48

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