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A Recursion Scheme for the Fisher Equation

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Abstract

In this paper, a perturbation method is used to approximate solution of the Fisher equation, assuming diffusion as a disturbance factor in the logistic equation. Approximate solution obtained depends on the initial value of ψ and its derivatives. The method given good result for initial data ψ bounded away from zero. For initial data close to zero at some point x, we can improve it by doing recursion with respect to t. This method is tested on a sample and the results show that the approximate solution can be improved by doing recursion.

Mathematics Subject Classification: 35K57, 35B25, 35A25

Keywords: Fisher equation; Perturbation method; recursion

1 Introduction

In 1937, Fisher proposed a nonlinear reaction–diffusion equation to describe the propagation of a viral mutant in an infinitely long habitat [4], it is given by

$$u_t = \varepsilon u_{xx} + \alpha u(1-u) \tag{1}$$

where $u(x,t)$ is the gene frequency at location x and time t. After rescaling the time variable $t^* = \alpha t$, spatial variable $x^* = \sqrt{\alpha}x$, and dropping the asterisk notation, (1) becomes

$$u_t = \varepsilon u_{xx} + u(1-u). \tag{2}$$

In this study, we will construct an approximation solution to the problem (2) with initial data ψ on the interval $[0,1] \subseteq \mathbb{R}$. Energy of solution of (2) at time $t > 0$, is defined as $E(t) = \int_0^1 |u_x|^2 dx$. It is used to indicate the accuracy of the approximation solution.

Numerous research has been carried out on Fisher and generalized Fisher equation. Some studied the numerical approach for its solution, e.g. Matinfar in [8] solved the Fisher equation via variational iteration method. Aminikhah in [1] proposed a combination of Laplace transform and homotopy perturbation methods to solve Fisher equations. Dattoli in [3] proposed a quasi-exact solution of the Fisher equation and Bastani in [2] presented sixth-order Compact Finite Difference (CFD6) scheme, a new numerical method to approximate the solution of the Fisher equations. Another direction of research is the energy decay. Soeharyadi in [11] investigated energy $E(t)$ for various value of ε, and showed a bound for energy, $E(t) \leq e^{2(1-\varepsilon\pi^2)t} E(0)$. As a corollary, $E(t)$ decays to zero as t goes to infinity, for sufficiently large diffusion factor. In this article, we will approach the Fisher equation solution for ε sufficiently small, using the perturbation method.

2 Method and discussion

Consider the following initial value problem,

$$\begin{aligned} u_t &= \varepsilon u_{xx} + u(1-u), \quad t > 0 \\ u(x,0) &= \psi(x). \end{aligned} \tag{3}$$

In this case, we will determine an approximate solution that does not depend on the boundary conditions.

In the absence of diffusion factor in (3) we have logistic equation, $u_t = u(1-u)$, with its exact solution

$$u(x,t) = \frac{\psi(x)}{\psi(x) + (1-\psi(x))e^{-t}}. \tag{4}$$

As t goes to infinity, the solution $u(x,t)$ tends to 1 pointwisely except when the initial data is zero.

For small ε, however, the general solution of (3) is no longer given by (4), and a correction terms must be added to it. We attempt a correction in the form

$$u(x,t;\varepsilon) = u_0(x,t) + \varepsilon u_1(x,t) + O(\varepsilon^2). \tag{5}$$

By substituting (5) into logistic diffusion equation in (3) and using additive property of derivative , we have

$$\begin{aligned} (u_0(x,t) + \varepsilon u_1(x,t) + O(\varepsilon^2))_t &= u_{0,t}(x,t) + \varepsilon u_{1,t}(x,t) \\ (u_0(x,t) + \varepsilon u_1(x,t) + O(\varepsilon^2))_{xx} &= u_{0,xx}(x,t) + \varepsilon u_{1,xx}(x,t). \end{aligned} \tag{6}$$

Subtituting (6) into logistic diffusion equation in (3) and collecting coefficients of equal power of ε we obtain

$$u_{0,t}(x,t) - u_0(x,t) + u_0^2(x,t) = 0 \tag{7}$$

and

$$u_{1,t}(x,t) - u_{0,xx}(x,t) - u_1(x,t) + 2u_0(x,t)u_1(x,t) = 0. \tag{8}$$

The initial data is

$$u_0(x,0) = \psi(x) \tag{9}$$

and

$$u_1(x,0) = 0 \tag{10}$$

Solution of (7) with initial data (9) is (4). To solve (8) with initial data (10), we define

$$F(x) = \frac{1-\psi(x)}{\psi(x)}, \tag{11}$$

and then we have

$$u_{0,xx} = \frac{e^{-t}(2F'(x)^2 e^{-t} - F''(x)(1+F(x)e^{-t}))}{(1+F(x)e^{-t})^3}, \tag{12}$$

where

$$F'(x) = \frac{-\psi'^2(x)}{\psi^2(x)},$$

and

$$F'(x) = \frac{2\psi'^2(x) - \psi''(x)\psi(x)}{\psi^3(x)}.$$

Solution of (8) is

$$u_1(x,t) = \frac{-e^{-t}}{1+F(x)e^{-t}}\left[\frac{2F'(x)^2}{F(x)} ln\left(\frac{1+F(x)e^{-t}}{1+F(x)}\right) + F''(x)t\right], \tag{13}$$

thus we obtain an approximation solution of initial value problem (3) is

$$\begin{aligned}\hat{u}(x,t;\varepsilon) &= \frac{\psi}{\psi+(1-\psi)e^{-t}} \\ &- \frac{\varepsilon e^{-t}}{\psi^3(\psi+(1-\psi)e^{-t})^2}\left[\frac{2\psi'^2}{1-\psi}\ln(\psi+(1-\psi)e^{-t}) + (2\psi'^2 - \psi''\psi)t\right]\end{aligned} \tag{14}$$

where $\psi = \psi(x)$. This approximation solution uses ψ and its first and second derivatives to predict the value of $u(x,t)$. The method given good result for initial data ψ far away from zero. At the points where the data is close to zero, we obtain poor results. To fix the situation, we apply recursions for (14). The Recursion process is done as follows:

1. Sets the recursion time t_R. To the best of our knowledge, there is no systematic method to determine the optimum value for recursion time $t_R > 0$. However, based on experiments on several initial conditions, the value of t_R is better when $\min_x \hat{u}(x, t_R) \approx 0.1$.

2. Sets $\hat{u}(x, t_R)$ as the initial data for the next approximation solution ($t > t_R$).

Thus, the approximation solution by recursion at time t_R is

$$\hat{u}_R(x, t; \varepsilon) = \begin{cases} \hat{u}(x,t) & t \le t_R, \\ \hat{w}(x,t) & t > t_R, \end{cases}$$

where $\hat{w}$ is approximation solution for the following initial value problem,

$$\begin{aligned} w_t &= \varepsilon w_{xx} + w(1-w), \quad t > 0 \\ w(x,0) &= \hat{u}(x, t_R). \end{aligned} \tag{15}$$

In the next section, some numerical examples are studied to demonstrate the accuracy of the proposed method.

3 An Illustration

In this section, an example are provided to illustrate the validity of the proposed method, for which the equation has an exact solution, so a comparative study can be made. We now consider the Fisher equation $u_t = \varepsilon u_{xx} + u(1-u)$ subject to the initial condition $\psi(x) = \frac{1}{(1+e^{x\sqrt{6\varepsilon}})^2}$ where Bastani in [2] proposed the exact solution as

$$u(x,t) = \frac{1}{(1 + e^{(x\sqrt{\frac{1}{6\varepsilon}} - \frac{5}{6}t)})^2}. \tag{16}$$

Comparisons are made with analytical solution and perturbation method for $\varepsilon = 0.1$ at time $t = 0.5$, 2, 4, 6, and $x = 0.25$, 0.50, 0.75 in Table 1. The perturbation and analytical solution is also depicted at different time levels in Figure 1.

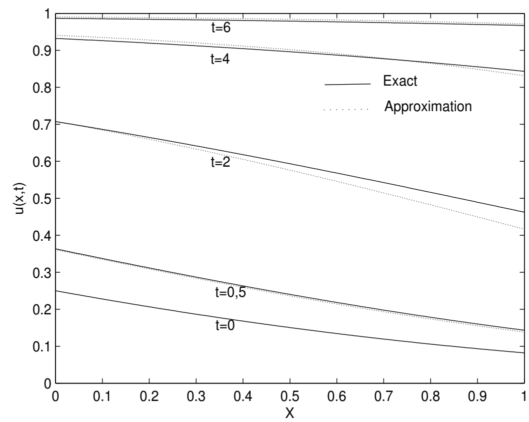

Figure 1: The graph of (14) and (16) with initial data $\psi(x) = \frac{1}{(1+e^{x\sqrt{6\varepsilon}})^2}$ and $\varepsilon = 0.1$ with no recursion.

t	x	Approximation	Exact	Absolute error
0.5	0.25	0.2961	0.2992	0.0030
	0.50	0.2362	0.2402	0.0040
	0.75	0.1832	0.1878	0.0046
2	0.25	0.6465	0.6532	0.0067
	0.50	0.5761	0.5934	0.0173
	0.75	0.4984	0.5293	0.0309
4	0.25	0.9242	0.9160	0.0082
	0.50	0.9019	0.8962	0.0057
	0.75	0.8718	0.8722	0.0004
6	0.25	0.9877	0.9833	0.0045
	0.50	0.9841	0.9791	0.0050
	0.75	0.9789	0.9738	0.0051

Table 1: Comparison of $\hat{u}$ with no recursion and exact solution.

Using recursion at $t = 0.5$, we obtain a better prediction. Comparison results using recursion are shown in Table 2, and the approximation and exact solution is depicted in Figure 2.

t	x	Approximation	Exact	Absolute error
0.5	0.25	0.2961	0.2992	0.0030
	0.50	0.2362	0.2402	0.0040
	0.75	0.1832	0.1878	0.0046
2	0.25	0.6499	0.6532	0.0033
	0.50	0.5840	0.5934	0.0094
	0.75	0.5115	0.5293	0.0178
4	0.25	0.9234	0.9160	0.0073
	0.50	0.9026	0.8962	0.0064
	0.75	0.8754	0.8722	0.0032
6	0.25	0.9872	0.9833	0.0039
	0.50	0.9837	0.9791	0.0046
	0.75	0.9789	0.9738	0.0051

Table 2: Comparison of $\hat{u}(x,t)$ with recursion at $t_R = 0.5$ and exact solution.

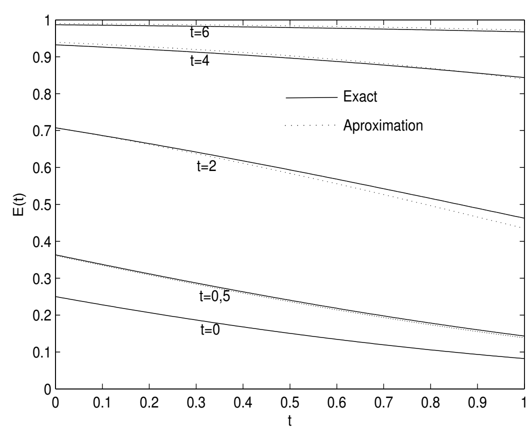

Figure 2: The graph of (14) and (16) with initial data $\psi(x) = \frac{1}{(1+e^{x\sqrt{6\varepsilon}})^2}$ and $\varepsilon = 0.1$ with recursion at $t_R = 0.5$.

Absolute error comparison of approximation with and without recursions at $t = 2$ and spatial step $\triangle x = 0.05$ is depicted in Figure (3). Total absolute error without recursion is 0.4075 while with recursion is 0.2321.

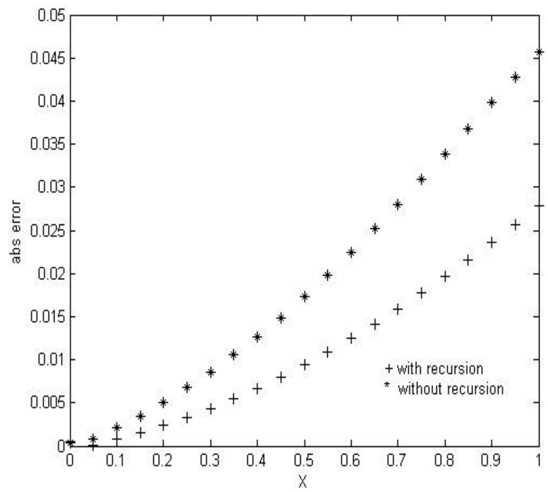

Figure 3: Error comparison between approximation with and without recursion at $t = 0.5$ when $t = 2$.

Comparison of energy between exact solution and approximation solution shows that, approximation solution with recursion gives better than approximation solution without recursion.

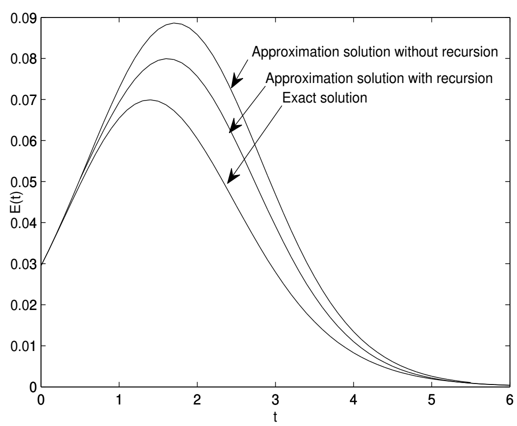

Figure 4: The graph of energy of exact and approximation solution with initial data is $\psi(x) = \frac{1}{(1+e^{x\sqrt{6\varepsilon}})^2}$ and $\varepsilon = 0.1$ with recursion at $t = 0.5$.

4 Conclusion

Fisher equation can be regarded as a logistic equation by assuming that the diffusion factor as perturbation factor. Perturbation method provides an approach that depends on the state of the initial value. If the initial value is close to zero, this approach gives a poor approximation, however we can improve the solution by doing recursion.

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