Remarks on the Hermite-Hadamard Type Inequalities for Harmonically Quasi-Convex Functions

Jaekeun Park

Department of Mathematics
Hanseo University
Seosan, Choongnam, 356-706, Korea

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Abstract

In this paper, some new results related to the right-hand side of the Hermite-Hadamard type inequality for the class of functions whose derivatives at certain powers are harmonically quasi-convex functions are obtained.

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1 Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard’s inequality, due to its rich geometrical significance and applications, which is stated as follows: Let $f : I \subseteq R \to R$ be a convex function and $a, b \in I$ with $a < b$. Then following double inequalities hold:

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \quad (1)$$
Hermite-Hadamard’s inequalities for convex, \((\alpha,m)\)-convex, \(GA\)-convex and geometric convex functions and have received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in \([1, 2, 3, 7, 8, 9, 10, 11, 13, 14]\) and references therein.

Let us recall some definitions of several kinds of convex functions:

**Definition 1.** Let \(I\) be an interval in \(R\). Then \(f: I \rightarrow R\) is said to be convex on \(I\) if the inequality
\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)
\]
holds, for all \(x, y \in I\) and \(t \in [0, 1]\).

**Definition 2.** Let \(I\) be an interval in \(R_+ = (0, \infty)\). A function \(f: I \rightarrow R\) is said to be harmonically convex on \(I\) if the inequality
\[
f\left(\frac{xy}{tx + (1 - t)y}\right) \leq tf(y) + (1 - t)f(x)
\]
holds, for all \(x, y \in I\) and \(t \in [0, 1]\). If the inequality in (2) is reversed, then \(f\) is said to be harmonically concave.

In [4], İmdat Işcan established the following result of the Hermite-Hadamard type for harmonically convex functions:

**Theorem 1.1.** Let \(f: I \subseteq R_+ = (0, \infty) \rightarrow R\) be a harmonically convex function on an interval \(I\) and \(f \in L[a,b]\), where \(a, b \in I\) with \(a < b\).

\[
f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \leq \frac{f(a) + f(b)}{2}.
\]

Also, in [4, 5, 6], İmdat Işcan established some new Hermite-Hadamard type and Ostrowski type inequalities, which estimate the difference between the middle and the rightmost terms in (3), for harmonically convex functions:

**Theorem 1.2.** Let \(f: I \subseteq R_+ = (0, \infty) \rightarrow R\) be a differentiable function on the interior \(I^0\) of an interval \(I\) in \(R_+ = (0, \infty)\) and \(f' \in L[a,b]\), where \(a, b \in I\) with \(a < b\). If \(|f'|^q\) is harmonically convex function on \([a,b]\) for \(q \geq 1\), then the following inequality holds:

\[
\left|\frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx\right|
\leq \frac{ab(b-a)}{2} \lambda_1^{\frac{1}{2}} \left[\lambda_2 \left|f'(a)\right|^q + \lambda_3 \left|f'(b)\right|^q\right]^{\frac{1}{q}},
\]

where \(\lambda_1, \lambda_2, \lambda_3\) are positive constants.
Hermite-Hadamard type inequalities

where

\[
\lambda_1 = \frac{1}{ab} - \frac{2}{(b - a)^2} \ln \left( \frac{(a + b)^2}{4ab} \right),
\]

\[
\lambda_2 = -\frac{1}{b(b - a)} + \frac{3a + b}{(b - a)^3} \ln \left( \frac{(a + b)^2}{4ab} \right),
\]

\[
\lambda_3 = \frac{1}{a(b - a)} - \frac{3b + a}{(b - a)^3} \ln \left( \frac{(a + b)^2}{4ab} \right)
= \lambda_1 - \lambda_2.
\]

In [15], Zhang et. al defined the harmonically quasi-convex function and supplied several properties of this kind of functions.

**Definition 3.** Let \( I \) be an interval in \( R_+ = (0, \infty) \). A function \( f : I \to R \) is said to be harmonically quasi-convex on \( I \) if the inequality

\[
f \left( \frac{xy}{tx + (1 - t)y} \right) \leq \sup \{ f(x), f(y) \}
\]

holds, for all \( x, y \in I \) and \( t \in [0, 1] \). If the inequality in (2) is reversed, then \( f \) is said to be harmonically quasi-concave.

In this article we consider the following special functions:

**Definition 4.** The hypergeometric function \( _2F_1[a, b, c, x] \) is defined for \(| x | < 1 \) by the power series

\[
_2F_1[a, b, c, x] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n.
\]

Here \((q)_n\) is the Pochhammer symbol, which is defined by

\[
(q)_n = \begin{cases} 
1, & n = 0 \\
q(q + 1) \cdots (q + n - 1), & n > 0.
\end{cases}
\]

In this paper, we give some new Hermite-Hadamard type inequalities, which gives an upper bound for the approximation of the integral average \( \frac{1}{b-a} \int_a^b f(u)du \) by the value \( \frac{f(a)+f(b)}{2} \), that is, estimate the difference between the middle and the rightmost terms in (1), for harmonically s-convex functions in the second sense by setting up an integral identity for differentiable functions.
2 Main results

In order to find some new inequalities of Hermite-Hadamard-like type inequalities connected with the rightmost and and middle parts of (1) for functions whose derivatives are harmonically s-convex in the second sense, we need the following lemma [12]:

**Lemma 1.** Let \( f : I \subseteq R_+ = (0, \infty) \rightarrow R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) such that \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). Then the following identity

\[
I_f(a, b; r) \equiv \frac{rf(a) + f(b)}{r + 1} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx
\]

\[
= \frac{ab(a - b)}{r + 1} \int_0^1 \frac{1 - (r + 1)t}{A_t^2(a, b)} f'(\frac{ab}{A_t(a, b)}) \, dt \quad (6)
\]

holds for \( r \in [0, 1] \), where \( A_t(a, b) = (1 - t)a + tb \).

**Proof** By the integration by parts, we have

\[
\int_0^1 \frac{1 - (r + 1)t}{A_t^2(a, b)} f'(\frac{ab}{A_t(a, b)}) \, dt
\]

\[
= \frac{1}{ab(b - a)} \left[ rf(a) + f(b) - (r + 1) \int_0^1 f\left(\frac{ab}{A_t(a, b)}\right) \, dt\right]
\]

\[
= \frac{1}{ab(b - a)} \left[ rf(a) + f(b) - (r + 1) \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx\right]
\]

which implies that the identity (6) holds.

Now we turn our attention to establish the Hermite-Hadamard type inequalities, which estimate the difference between the middle and the leftmost terms in (1), for harmonically quasi-convex functions in the second sense by using the above lemma.

**Theorem 2.1.** Let \( f : I \subseteq R_+ = (0, \infty) \rightarrow R \) be a differentiable function on \( I^0 \), the interior of a interval \( I \), such that \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'| \) is harmonically quasi-convex on \([a, b]\), then for all \( t \in [0, 1] \) the following inequality holds:

\[
|I_f(a, b; r)| \leq \left\{ \frac{b - ra}{r + 1} + \frac{ab}{b - a} \ln\left[\frac{ab(r + 1)^2}{(ra + b)^2}\right]\right\}
\]

\[
\times \sup \left\{ |f'(a)|, |f'(b)| \right\} \quad (7)
\]
Proof. From Lemma 1, we have

\[
|I_f(a, b; r)| \leq \frac{ab(b - a)}{r + 1} \int_0^1 |1 - (r + 1)t| \left| f'\left( \frac{ab}{A_t(a, b)} \right) \right| dt
\]

\[
= \frac{ab(b - a)}{r + 1} \left[ \int_0^{\frac{r+1}{r+1}} \frac{1 - (r + 1)t}{A_t^2(a, b)} \left| f'\left( \frac{ab}{A_t(a, b)} \right) \right| dt + \int_{\frac{r+1}{r+1}}^1 \frac{(r + 1)t - 1}{A_t^2(a, b)} \left| f'\left( \frac{ab}{A_t(a, b)} \right) \right| dt \right].
\]

Since \(|f'|^q\) is harmonically quasi-convex on \([a, b]\), we have

\[
|I_f(a, b; r)| \leq \frac{ab(b - a)}{r + 1} \left\{ \int_0^{\frac{r+1}{r+1}} \frac{1 - (r + 1)t}{A_t^2(a, b)} dt + \int_{\frac{r+1}{r+1}}^1 \frac{(r + 1)t - 1}{A_t^2(a, b)} dt \right\}
\]

\[
\times \sup \left\{ \left| f'(a) \right|, \left| f'(b) \right| \right\}
\]

\[
= \left\{ \frac{b - ra}{r + 1} + \frac{ab}{b - a} \ln \left[ \frac{ab(r + 1)}{(ra + b)^2} \right] \right\} \sup \left\{ \left| f'(a) \right|, \left| f'(b) \right| \right\},
\]

where we have used the facts that

\[
(i) \int_0^{\frac{r+1}{r+1}} \frac{1 - (r + 1)t}{A_t^2(a, b)} dt = \frac{1}{a(b - a)} + \frac{r + 1}{(b - a)^2} \ln \left[ \frac{a(r + 1)}{ra + b} \right],
\]

\[
(ii) \int_{\frac{r+1}{r+1}}^1 \frac{(r + 1)t - 1}{A_t^2(a, b)} dt = -\frac{r}{b(b - a)} + \frac{r + 1}{(b - a)^2} \ln \left[ \frac{b(r + 1)}{ra + b} \right].
\]

Therefore, we can deduce the following results:

**Corollary 2.1.** Let \( f : I \subseteq R_+ = (0, \infty) \to R \) be a differentiable function on \( I^0 \), the interior of an interval \( I \), such that \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). Assume \(|f'|^q\) is harmonically quasi-convex on \([a, b]\).

(1) If \( r = 1 \) in (7), then the following inequality holds:

\[
|I_f(a, b; 1)| \leq \left\{ \frac{b - a}{2} + \frac{ab}{b - a} \ln \left[ \frac{4ab}{(a + b)^2} \right] \right\} \sup \left\{ \left| f'(a) \right|, \left| f'(b) \right| \right\}.
\]
(2) If \( r = 0 \) in (7), then the following inequality holds:

\[
|I_f(a, b; 0)| = \left| f(b) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right|
\leq \left\{ b + \frac{ab}{b-a} \ln \left[ \frac{a}{b} \right] \right\} \sup \left\{ |f'(a)|, |f'(b)| \right\}.
\]

**Theorem 2.2.** Let \( f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^0 \), the interior of an interval \( I \), such that \( f' \in L([a,b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is harmonically quasi-convex on \([a, b]\) for \( q \geq 1 \), then for all \( x \in [a, b] \) the following inequality holds:

\[
|I_f(a, b; r)| \leq \frac{ab(b-a)}{(r+1)^{1+\frac{1}{p}}} \left\{ \mu_{21}(a, b, r, q) + \frac{1}{r+1} \mu_{22}(a, b, r, q) \right\}
\times \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}, \quad (8)
\]

where

\[
\mu_{21}(a, b, r, q) = \frac{1}{(1+q)(1+r)a^{2q}} \quad 2F_1[1, 2q, 2 + q, -\frac{b-a}{(1+r)a}],
\]

\[
\mu_{22}(a, b, r, q) = \frac{r^{1+q}(1+r)^{2q-1}}{(1+q)(ra+b)^{2q}} \quad 2F_1[1 + q, 2q, 2 + q, -\frac{r(b-a)}{ra+b}].
\]

**Proof.** From Lemma 1, we have

\[
|I_f(a, b; r)|
\leq \frac{ab(b-a)}{r+1} \left[ \int_0^{\frac{1}{r+1}} \frac{1 - (r+1)t}{A^2_t(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right| \, dt \right.
\]

\[
+ \int_{\frac{1}{r+1}}^1 \frac{(r+1)t - 1}{A^2_t(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right| \, dt \bigg].
\]

By the harmonically quasi-convexity of \( |f'|^q \) and using the Hölder integral
inequality, we have
\[
\left| I_f(a, b; r) \right| \leq \frac{ab(b-a)}{r+1} \left\{ \left( \int_0^{\frac{1}{r+1}} 1 \, dt \right)^{\frac{1}{q}} \left( \int_0^{\frac{1}{r+1}} \frac{1-(r+1)t}{A^q_t(a, b)} \, dt \right)^{\frac{1}{r}} + \left( \int_0^{\frac{1}{r+1}} 1 \, dt \right)^{\frac{1}{r}} \left( \int_0^{\frac{1}{r+1}} \frac{(r+1)t-1}{A^q_t(a, b)} \, dt \right)^{\frac{1}{q}} \right\}
\]
\[
= \frac{ab(b-a)}{(r+1)^{1+\frac{1}{r}} + \frac{1}{r}} \left\{ \left( \int_0^{\frac{1}{r+1}} \frac{1-(r+1)t}{A^q_t(a, b)} \, dt \right)^{\frac{1}{r}} + r^{\frac{1}{q}} \left( \int_0^{\frac{1}{r+1}} \frac{(r+1)t-1}{A^q_t(a, b)} \, dt \right)^{\frac{1}{r}} \right\} \left( \sup \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}}
\]
which completes the proof.

**Corollary 2.2.** In the inequality (8) in Theorem 2.2, additionally, if \(| f'(x) | \leq M \) for \( x \in [a, b] \), then the following inequality holds:
\[
\left| I_f(a, b; r) \right| \leq \frac{ab(b-a)M}{(r+1)^{1+\frac{1}{r}}} \left\{ \mu^q_{21}(a, b, r, q) + r^{\frac{1}{q}} \mu^q_{22}(a, b, r, q) \right\}.
\]

**Theorem 2.3.** Let \( f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^0 \), the interior of an interval \( I \), such that \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). If \(| f'|^q \) is harmonically quasi-convex on \([a, b]\) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then for all \( x \in [a, b] \) the following inequality holds:
\[
\left| I_f(a, b; r) \right| \leq \frac{ab(b-a)}{(r+1)^{1+\frac{1}{r}}} \left\{ \frac{b-ra}{ab(b-a)} + \frac{r+1}{(b-a)^2} \ln \left| \frac{ab(r+1)^2}{(ra+b)^2} \right| \right\} \left( \sup \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}}.
\]

**Proof.** From Lemma 1, we have
\[
\left| I_f(a, b; r) \right| \leq \frac{ab(b-a)}{r+1} \left[ \int_0^{\frac{1}{r+1}} \frac{1-(r+1)t}{A^q_t(a, b)} \, dt \left| f' \left( \frac{ab}{A^q_t(a, b)} \right) \right| \right] + \int_0^{\frac{1}{r+1}} \frac{(r+1)t-1}{A^q_t(a, b)} \, dt \left| f' \left( \frac{ab}{A^q_t(a, b)} \right) \right| dt.
\]
By the harmonically quasi-convexity of $|f'|^q$ and using the Hölder integral inequality, we have

$$
\left| I_f(a, b; r) \right| \leq \frac{ab(b - a)}{r + 1} \left[ \left( \int_0^{\frac{1}{r+1}} \frac{1 - (r + 1)t}{A_t^2(a, b)} \ dt \right)^{\frac{1}{q}} \left( \int_0^{\frac{1}{r+1}} \frac{1 - (r + 1)t}{A_t^2(a, b)} \ dt \right)^{\frac{1}{p}} \right]^{\frac{1}{q}}
$$

$$
+ \left( \int_{\frac{1}{r+1}}^{1} \frac{(r + 1)t - 1}{A_t^2(a, b)} \ dt \right)^{\frac{1}{p}} \left( \int_{\frac{1}{r+1}}^{1} \frac{(r + 1)t - 1}{A_t^2(a, b)} \ dt \right)^{\frac{1}{q}}
$$

$$
\times \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}
$$

$$
= \frac{ab(b - a)}{r + 1} \left\{ \frac{b - ra}{ab(b - a)} + \frac{r + 1}{(b - a)^2} \ln \left[ \frac{ab(r + 1)^2}{(ra + b)^2} \right] \right\}
$$

$$
\times \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}},
$$

which completes the proof.

**Theorem 2.4.** Let $f : I \subseteq \mathbb{R}^+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^0$, the interior of an interval $I$, such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a, b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for all $x \in [a, b]$ the following inequality holds:

$$
\left| I_f(a, b; r) \right| \leq \frac{ab(b - a)}{r + 1} \left[ \left( \int_0^{\frac{1}{r+1}} \frac{1 - (r + 1)t}{A_t^2(a, b)} \ dt \right)^{\frac{1}{q}} \left( \int_0^{\frac{1}{r+1}} \frac{1 - (r + 1)t}{A_t^2(a, b)} \ dt \right)^{\frac{1}{p}} \right]^{\frac{1}{q}}
$$

$$
\times \left[ \mu_{41}(a, b, r, q) + r^{1+\frac{q}{p}} \mu_{42}(a, b, r, q) \right]
$$

$$
\times \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}},
$$

where

$$
\mu_{41}(a, b, r, q) = \{a(1 + r)\}^{1-2q} - (ra + b)^{1-2q},
$$

$$
\mu_{42}(a, b, r, q) = (ra + b)^{1-2q} - \{b(1 + r)\}^{1-2q}.
$$

**Proof.** From Lemma 1 and the Hölder integral inequality, we have

$$
\left| I_f(a, b; r) \right| \leq \frac{ab(b - a)}{r + 1} \left[ \left( \int_0^{\frac{1}{r+1}} \frac{1 - (r + 1)t}{A_t^2(a, b)} \ dt \right) \left| f'\left( \frac{ab}{A_t(a, b)} \right) \right| dt \right]
$$

$$
+ \left( \int_{\frac{1}{r+1}}^{1} \frac{(r + 1)t - 1}{A_t^2(a, b)} \ dt \right) \left| f'\left( \frac{ab}{A_t(a, b)} \right) \right| dt \right].
$$
By the harmonically quasi-convexity of $|f'|^q$ and using the Hölder integral inequality, we have

$$\left| I_f(a, b; r) \right| \leq \frac{ab(b - a)}{r + 1} \left[ \left( \int_0^{\frac{1}{r+1}} \{1 - (r + 1)t\}^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{r+1}} \frac{1}{A_i^q(a, b)} dt \right)^{\frac{1}{q}} + \left( \int_1^{\frac{1}{r+1}} \{r + 1\}^p dt \right)^{\frac{1}{p}} \left( \int_1^{\frac{1}{r+1}} \frac{1}{A_i^q(a, b)} dt \right)^{\frac{1}{q}} \right] \times \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \leq \frac{ab(b - a)}{r + 1} \left[ \left( \int_0^{\frac{1}{r+1}} \{1 - (r + 1)t\}^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{r+1}} \frac{1}{A_i^q(a, b)} dt \right)^{\frac{1}{q}} + \left( \int_1^{\frac{1}{r+1}} \{r + 1\}^p dt \right)^{\frac{1}{p}} \left( \int_1^{\frac{1}{r+1}} \frac{1}{A_i^q(a, b)} dt \right)^{\frac{1}{q}} \right] \times \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \times \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \times \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}},$$

where we have used the fact that

$$\int_0^{\frac{1}{r+1}} \frac{1}{A_i^q(a, b)} dt = \frac{a(1 + r)^{1-2q} - (ra + b)^{1-2q}}{(2q - 1)(1 + r)^{1-2q}(b - a)},$$

$$\int_1^{\frac{1}{r+1}} \frac{1}{A_i^q(a, b)} dt = \frac{(ra + b)^{1-2q} - b(1 + r)^{1-2q}}{(2q - 1)(1 + r)^{1-2q}(b - a)},$$

$$\int_0^{\frac{1}{r+1}} \{1 - (r + 1)t\}^p dt = \frac{1}{(1 + p)(1 + r)},$$

$$\int_1^{\frac{1}{r+1}} \{r + 1\}^p dt = \frac{1}{(1 + p)(1 + r)}.$$

**Theorem 2.5.** Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a differentiable function on $I^0$, the interior of an interval $I$, such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a, b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for all $x \in [a, b]$ the following inequality holds:

$$\left| I_f(a, b; r) \right| \leq ab(b - a) \left( \frac{1}{1 + q} \right)^{\frac{1}{q}} \left( \frac{1}{2p - 1} \right)^{\frac{1}{p}} \times \left[ \mu_{41}^p(a, b, r, p) + r^{1+\frac{1}{p}} \mu_{42}^p(a, b, r, p) \right] \times \left( \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}},$$

where $\mu_{4i}(i = 1, 2)$ are defined in Theorem 2.4.
Proof. From Lemma 1 and the Hölder integral inequality, we have

\[
|I_f(a, b; r)| \leq \frac{ab(b - a)}{r + 1} \left[ \int_0^\frac{r+1}{r+1} \frac{1 - (r + 1)t}{A_t^2(a, b)} \left| f'(\frac{ab}{A_t(a, b)}) \right| dt \right.
+ \int_\frac{r+1}{r+1}^1 \frac{(r + 1)t - 1}{A_t^2(a, b)} \left| f'(\frac{ab}{A_t(a, b)}) \right| dt].
\]

By the harmonically quasi-convexity of $|f'|^q$ and using the Hölder integral inequality, we have

\[
|I_f(a, b; r)| \leq \frac{ab(b - a)}{r + 1} \left( \sup \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}}
\times \left[ \left( \int_0^\frac{r+1}{r+1} \frac{1}{A_t^2(a, b)} dt \right)^{\frac{1}{p}} \left( \int_0^\frac{r+1}{r+1} \{1 - (r + 1)t\}^q dt \right)^{\frac{1}{q}}
+ \left( \int_\frac{r+1}{r+1}^1 \frac{1}{A_t^2(a, b)} dt \right)^{\frac{1}{p}} \left( \int_\frac{r+1}{r+1}^1 \{(r + 1)t - 1\}^q dt \right)^{\frac{1}{q}} \right]
\leq ab(b - a)^{\frac{1}{q}} \left( \frac{1}{1 + q} \right)^{\frac{1}{2}} \left( \frac{1}{2p - 1} \right)^{\frac{1}{2}} \left\{ \mu_{21}^q(a, b, r, p) + \mu_{22}^q(a, b, r, p) \right\}
\times \left( \sup \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}},
\]

which completes the proof.

Theorem 2.6. Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I^0$, the interior of an interval $I$, such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a, b]$ for $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for all $x \in [a, b]$ the following inequality holds:

\[
|I_f(a, b; r)| \leq \frac{ab(b - a)}{(r + 1)^{1 + \frac{1}{q}} \left\{ \mu_{21}^q(a, b, r, p) + r^{\frac{1}{2}} \mu_{22}^q(a, b, r, p) \right\}}
\times \left( \sup \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}}.
\]

Proof. From Lemma 1, Hölder integral inequality and the harmonically
Hermite-Hadamard type inequalities

quasi-convexity of \(|f'|^q\), we have

\[
\begin{align*}
|I_f(a, b; r)| & \leq \frac{ab(b - a)}{r + 1} \left[ \int_0^{\frac{1}{r + 1}} 1 - (r + 1)t \frac{1}{A_t^2(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right| dt \\
& \quad + \int_1^{\frac{1}{r + 1}} (r + 1)t - 1 \frac{1}{A_t^2(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right| dt \right] \\
& \leq \frac{ab(b - a)}{r + 1} \\
& \times \left[ \left( \int_0^{\frac{1}{r + 1}} \frac{1 - (r + 1)t}{A_t^{2p}(a, b)} dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{r + 1}} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right| dt \right)^{\frac{1}{q}} \\
& \quad + \left( \int_1^{\frac{1}{r + 1}} \frac{(r + 1)t - 1}{A_t^{2p}(a, b)} dt \right)^{\frac{1}{p}} \left( \int_1^{\frac{1}{r + 1}} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right| dt \right)^{\frac{1}{q}} \right] \\
& \leq \frac{ab(b - a)}{(r + 1)^{1 + \frac{1}{q}}} \left\{ \mu_{21}^\frac{1}{p}(a, b, r, p) + r^\frac{1}{p} \mu_{22}^\frac{1}{p}(a, b, r, p) \right\} \\
& \quad \times \left( \sup \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right)^{\frac{1}{q}},
\end{align*}
\]

which completes the proof.

References


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