Landau-Silin Kinetic Equation in the Theory of Normal Fermi Liquid

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Abstract

In the frame of the Kadanoff-Baym version of the Green’s function method in quantum statistical mechanics it is proved that the Landau-Silin kinetic equation for a normal quantum Fermi liquid is valid in the case when the widths of one-particle energy levels are taken into account exactly.

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1 Introduction

The problem of deriving quantum kinetic equations during decades determined one of the main directions of many-body system theory’s development. Special attention was attracted to transport phenomena in quantum liquids both in the frame of phenomenological and microscopic theories. Landau-Silin (LS) kinetic equation in the theory of normal Fermi liquid proved to be an effective tool for describing non-equilibrium properties of neutral and charged Fermi liquids [12]. Landau offered a kinetic equation for a neutral quasiparticle distribution function in Wigner’s semiclassical approach which may be applied to a macroscopic perturbation, the scale of which is large compared to the atomic scale. Silin has first shown that the equation may be also applied to the systems of charged particles provided the difficulties associated with the long range of the Coulomb interaction between the particles are removed if we allow for the dynamic screening of the particle motion in self-consistent fashion. The experimental discovery of the phenomena, theoretically predicted on the basis of the LS theory (zero sound in He-3 and spin waves in non-ferromagnetic metals) and the excellent agreement of the experimental data with the theoretical predictions made this phenomenological theory a subject of investigation on the basis of strict fundamental many-body theory.

The most convenient and effective approach to the problem is based on the real-time Green’s function formalism of Martin and Schwinger, further developed by Kadanoff and Baym (KB) [6]. It’s likely to state that the KB approach developed for equilibrium and non-equilibrium problems and for zero and finite temperatures is capable of describing both the statistics and dynamics of the many-body systems in a comprehensive way. A brief review of numerous various developments of the KB method and its applications to different many-body systems may be found in [2]. In this paper we will concentrate on the problem of rigorous derivation of the LS kinetic equation.

The first microscopic derivation of the LS equation was presented in [6]. The keystone of the derivation was a smallness of the widths of one-particle energy states near the Fermi level. This approximation involves an assumption about the continuity of the self-energy function at the Fermi level. The continuity can be proved in all orders of perturbation theory, but it is not necessarily true for situations, such as the superfluid and superconducting states, in which perturbation theory is not valid. Therefore, the presented in [6] derivation of the
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LS kinetic equation applies only to the so-called “normal” fermion systems at absolute zero temperature. Unfortunately, the assumptions are not valid for real fermion systems. We will show that these assumptions are only sufficient, but not necessary conditions for the validity of the LS equation, which can be proved for much more soft conditions.

Experimental discovery of a superfluidity of He-3 at low, but finite temperature, initiated a wave of attempts to derive a kinetic or transport equations for a normal Fermi liquid that would take into account finite widths of one-particle energy levels and would be valid at finite temperature [2,7-10,14]. Main difficulty in deriving the LS kinetic equation was associated with the necessity of a mathematically lawful elimination of second Poisson bracket in the right side of the KB generalized quantum kinetic equation (see below). The failure of the attempts was determined by the usage of wrong approximations for the spectral functions which didn’t satisfy the KB equation for the spectral function in the case of slightly non-equilibrium systems. The approximation for the spectral function, which satisfies the KB equation, was offered in [2] on the basis of certain relations in the theory of the Fourier transform, but the problem of the convergence of the appearing expansion was not considered. Thus, the validity of the LS kinetic equation was proved in [2] only with the precision up to the linear terms in the widths of one-particle energy levels. The convergence of the expansion of the spectral function was considered in [5], what made possible the rigorous derivation of the LS equation, exactly taking into account the widths of energy levels.

We will show that the LS kinetic equation in the case of slowly varying in space and time disturbances takes into account the widths of energy levels exactly. The following section contains a short presentation of the necessary formulas of KB theory which will be used for the analysis of the expansion for the spectral function and for the derivation of the LS equation. In Sec. 3 a discussion and summary are presented.

2 Spectral function and kinetic equation in Kadanoff-Baym formalism

The KB formalism leads to the following general expression for the one-particle spectral function $a (\vec{p}, \omega)$ of a system in equilibrium [6]:
\[ a(\tilde{p},\omega) = \frac{\Gamma(\tilde{p},\omega)}{(\omega - e(\tilde{p},\omega))^2 + \Gamma^2(\tilde{p},\omega)/4}, \]  

(1)

where

\[ e(\tilde{p},\omega) = E^{\text{HF}}(\tilde{p}) + \text{Re} \sum_c(\tilde{p},\omega), \]  

(2)

and \( E^{\text{HF}}(\tilde{p}) \) is a one-particle energy in the Hartree-Fock approximation. Real and imaginary \( (\Gamma(\tilde{p},\omega)) \) parts of the correlation self-energy function \( \sum_c(\tilde{p},\omega) \) are related to each other through the Hilbert transform:

\[ \text{Re} \sum_c(\tilde{p},\omega) = P \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\Gamma(\tilde{p},\omega)}{\omega - \omega} \]  

(3)

Here \( P \) refers to a principal value integration.

In the case of slowly varying in space and time disturbances, after the transition to the Wigner coordinates

\[ \tilde{R} = (\tilde{r}_i + \tilde{r}_r)/2, \quad \tilde{r} = \tilde{r}_i - \tilde{r}_r, \quad T = (t_i + t_r)/2, \quad t = t_i - t_r \]  

(4)

and the performance of the Fourier transform with respect to \( \tilde{r} \) and \( t \), all the quantities entering the theory are considered to be the functions of \( \tilde{p},\omega,\tilde{R},T \). For example, \( a = a(\tilde{p},\omega,\tilde{R},T) \).

The functions \( E^{\text{HF}} \) and \( \text{Re} \sum_c \) include the interaction with the external field.

If we take into account only the first derivatives with respect to slowly varying quantities \( \tilde{R} \) and \( T \) in the KB equations for the correlation functions, we come to the following equation for the spectral function \( a(\tilde{p},\omega,\tilde{R},T) \):

\[ \left[\omega - e(\tilde{p},\omega,\tilde{R},T), a(\tilde{p},\omega,\tilde{R},T)\right] + \left[\text{Re}g(\tilde{p},\omega,\tilde{R},T), \Gamma(\tilde{p},\omega,\til{R},T)\right] = 0, \]  

(5)

and to the generalized KB kinetic equation for the correlation function \( g^<(\tilde{p},\omega,\til{R},T) \).
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\[ \omega - e(\tilde{p} \omega; \tilde{R}T), g^c(\tilde{p} \omega; \tilde{R}T) + \left[ \text{Re} g(\tilde{p} \omega; \tilde{R}T), \sum \zeta^e(\tilde{p} \omega; \tilde{R}T) \right] = \left( \sum \zeta^c - \sum \zeta^c \right)(\tilde{p} \omega; \tilde{R}T) \] (6)

Here \([A,B]\) is the generalized Poisson bracket defined by the expression:

\[ [A,B] = \frac{\partial A}{\partial \omega} \frac{\partial B}{\partial T} - \frac{\partial A}{\partial T} \frac{\partial B}{\partial \omega} - \frac{\partial A}{\partial \tilde{p}} \frac{\partial B}{\partial \tilde{R}} + \frac{\partial A}{\partial \tilde{R}} \frac{\partial B}{\partial \tilde{p}}. \] (7)

The exact solution of Eq. (5) is given by the expression [6]:

\[ g(\tilde{p} z; \tilde{R}T) = \left( z - E^{HF}(\tilde{p}; \tilde{R}T) - \text{Re} \sum (\tilde{p} z; \tilde{R}T) \right)^{-1} \] (8)

In fact, the solution (8) leads to almost the same evaluation of the spectral function \( a(\tilde{p} \omega; \tilde{R}T) \) as in the equilibrium state:

\[ a(\tilde{p} \omega; \tilde{R}T) = \frac{\Gamma(\tilde{p} \omega; \tilde{R}T)}{(\omega - e(\tilde{p} \omega; \tilde{R}T))^2 + \Gamma^2(\tilde{p} \omega; \tilde{R}T)/4} \] (9)

Eq. (6) provides an exact description of the response to slowly varying disturbance. All the quantities appearing in this equation may be expressed in terms of correlation and self-energy functions. The result (9) means that in the case of slowly varying disturbances the approximations for the non-equilibrium spectral function may be written in the same form as in the equilibrium case. In order to simplify the notation, we will omit space and time variables.

In the KB formalism the quasiparticle energy \( E = E(\tilde{p}) \) is defined as a solution of the equation [6]:

\[ E(\tilde{p}) = e(\tilde{p}, \omega) \bigg|_{\omega = E(\tilde{p})} \] (10)

Now we expand \( e(\tilde{p}, \omega) \) as a function of \( \omega \) in Taylor series near the value \( E(\tilde{p}) \) and save only linear terms:

\[ e(\tilde{p}, \omega) = E(\tilde{p}) + \left. \frac{\partial e(\tilde{p}, \omega)}{\partial \omega} \right|_{\omega = E(\tilde{p})} (\omega - E(\tilde{p})) \] (11)
We substitute (11) into Eq. (1) and get the formula for $a_{QP}$ called the spectral function of the quasiparticle state:

$$ a_{QP}(\tilde{p}, \omega) = \frac{Z^2 \Gamma(\tilde{p}, E(\tilde{p}))}{(\omega - E(\tilde{p}))^2 + Z^2 \Gamma^2(\tilde{p}, E(\tilde{p})) / 4} \tag{12} $$

where $Z = Z(\tilde{p})$ is a renormalizing factor defined by the expression

$$ Z^{-1} = 1 - \left. \frac{\partial e(\tilde{p}, \omega)}{\partial \omega} \right|_{\omega = E(\tilde{p})} \tag{13} $$

It is easy to prove that $Z < 1$ for all values of $\tilde{p}$ when correlation energy $\sum \epsilon$ is taken into account. The quantity $\gamma = Z \Gamma$ can be considered the width of the quasiparticle energy level, and Eq. (12) for the spectral function may be rewritten as

$$ a_{QP}(\tilde{p}, \omega) = Z \frac{\gamma}{(\omega - E(\tilde{p}))^2 + \gamma^2 / 4} \tag{14} $$

The spectral function (1) or (9) may be presented in the form of the expansion in power series of the width of energy levels with a help of the following relation [2]:

$$ \int_{-\infty}^{\infty} dt \exp(-\frac{\Gamma}{2}|t|) \exp(itx) = \frac{\Gamma}{x^2 + \Gamma^2 / 4}, \quad \Gamma > 0 \tag{15} $$

The expansion of the first exponent in the left side of (15) in Taylor series in powers of $\Gamma$ with the subsequent term-by-term integration with a help of the formulas for the Fourier transforms of powers of $|t|$ allows to generalize the results of [6] and to get the expansion of the spectral function $a(\tilde{p}, \omega)$ in powers of $\Gamma$. These formulas may be presented in the form [4]:

$$ \int_{-\infty}^{\infty} t^{2n} \exp(itx) \ dt = 2\pi(-i)^{2n} \delta^{(2n)}(x), \quad n = 0, 1, 2, \ldots \tag{16} $$
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\[ \int_{-\infty}^{\infty} \exp(itx) \ dt = -2\sin \left( (2n+1) \frac{\pi}{2} \right) \left( 2n+1 \right) \frac{1}{\sin^2 x}, \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (17)

The terms of the expansion containing the even powers of the energy level's width can be considered on the basis of the formula [1,13]

\[ x\delta'(x) = -\delta(x) \]  \hspace{1cm} (18)

It is easy to prove by mathematical induction that

\[ \delta^{(n)}(x) = \frac{(-1)^n n!}{x^n} \delta(x) \]  \hspace{1cm} (19)

In correspondence with (16) and (19) the terms with even powers of \( \Gamma \) generate a geometric sequence, and the sum of the sequence is equal to [5]

\[ a_{\text{even}}(\tilde{p}, \omega) = 2\pi \frac{(\omega - e(\tilde{p}, \omega))^2}{(\omega - e(\tilde{p}, \omega))^2 + \Gamma^2 (\tilde{p}, \omega) / 4} \delta(\omega - e(\tilde{p}, \omega)) \]  \hspace{1cm} (20)

Due to the presence of the Dirac delta function in the numerator it is clear that the contribution of the sum (20) to all expressions containing integration with respect to frequency variable equals zero.

The terms with odd powers generate a geometric sequence with the same common ratio, and its sum equals to the complete expression (1) or (9) for the spectral function. Taking into account the remark after (20), it is convenient to write it in the form of the expansion

\[ a(\tilde{p}, \omega) = 2\pi \delta(\omega - e) + \frac{\Gamma}{(\omega - e)^2 - \frac{\Gamma^3}{4(\omega - e)^4} + \frac{\Gamma^5}{16(\omega - e)^6} - \ldots \ldots} \]  \hspace{1cm} (21)

The expansion for the real part of the Green function \( \text{Re } g \) with a help of (8) can be presented in the form

\[ \text{Re } g(\tilde{p}, \omega) = \frac{\omega - e}{(\omega - e)^2 + \Gamma^2 / 4} = \frac{1}{\omega - e} - \frac{\Gamma^2}{4(\omega - e)^3} + \frac{\Gamma^4}{16(\omega - e)^5} - \ldots \ldots \]  \hspace{1cm} (22)
These statements generalize the results obtained in [2] and open a way for the rigorous deriving the LS equation. Main problems here are associated with the necessity of the elimination of the second Poisson bracket in the left side of Eq. (6) in a mathematically lawful way. The reason of the failure of the attempts of such elimination, for example in [14], was the usage of an improper approximation to the spectral function. A special name “puzzling term” for the second Poisson brackets in Eqs. (5) and (6) was offered in [14] after such useless attempts. The proof of the validity of the LS kinetic equation may be produced in the following way. The first term in the expansion (21) for the spectral function after the substitution to the first Poisson bracket in Eq. (7) leads to the LS kinetic equation after integration with respect to frequency [6]. This equation may be written in the form:

$$\frac{\partial n}{\partial T} + \frac{\partial E}{\partial \tilde{p}} \cdot \frac{\partial n}{\partial \tilde{R}} - \frac{\partial E}{\partial \tilde{R}} \cdot \frac{\partial n}{\partial \tilde{p}} = I_{\text{collision}},$$

(23)

where \( n \) is the quasiparticle distribution function defined by the expression:

$$n(\tilde{p};\tilde{R}T) = f(\tilde{p} \omega;\tilde{R}T)\bigg|_{\omega = E(\tilde{p},\tilde{R})}$$

(24)

Due to the property (20) which eliminates the “extra” terms with even powers of \( \Gamma \) in the expansion for the spectral function, we see that there appear the “couples” of the terms from the expansions (21) and (22) which, being substituted to the first and second Poisson brackets correspondingly in the left sides of Eqs. (5) and (6), cancel each other. Indeed, the corresponding “couples” of the terms are:

$$(-1)^n \frac{\Gamma^{2n+1}}{4^n (\omega - e)^{2n+2}}$$

(25)

in the expansion (21), and

$$(-1)^n \frac{\Gamma^{2n}}{4^n (\omega - e)^{2n+1}}$$

(26)

in the expansion (22). These terms make the following contributions to the mentioned Poisson brackets:
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\[ [\omega - e, \alpha] \rightarrow (-\frac{1}{4})^n (2n + 1) \Gamma^{2n} [\omega - e, \Gamma] \]  \hspace{1cm} (27)

and

\[ \text{Re} g, \Gamma \rightarrow -(-\frac{1}{4})^n (2n + 1) \Gamma^{2n} [\omega - e, \Gamma] \]  \hspace{1cm} (28)

Thus, we see that only the first term in the expansion (21) for the spectral function makes the contribution to the left side of the kinetic equation (23).

3 Conclusions

Intuitive considerations demand that in case of slowly varying in space and time disturbances the kinetic equation for the distribution function should have the form (23), whatever complex the energies \( E(p,R,T) \) could be. But the question arises, if a strong interaction between the particles of a system may be taken into account in terms of such one-particle energies. We have proved that in the cases, when dynamic treatment of a system under consideration is possible, the kinetic equation for the quasiparticle distribution function always has a form of LS equation provided there is one-to-one correspondence between the energies of quasiparticles and one-particle energy states in the system. The widths of the one-particle energy levels do not influence the form of the left side of kinetic equation, that describes the kinetic effects of the potential, i.e., how the potential changes the energy-momentum relation from that of free particles to the more complex spectrum. The right side of the kinetic equation contains collision integrals that describe the dynamical effect of collisions. The form of these integrals may be affected by the widths of the energy levels.

Thus, the general picture of the behavior of an arbitrary fermion system can be the following. At zero or very close to zero temperature a system can reveal superfluid or superconducting behavior. In this situation there is no one-to-one correspondence between one-particle energy levels and “complex” quasiparticles (for example, Cooper pairs) and the LS equation can’t be used for the description of transport phenomena. As the temperature arises, the complex quasiparticle
states are destroyed, and one-particle energy spectrum is restored. In this situation the usage of the LS kinetic equation becomes lawful.

The LS kinetic equation allows to analyze the collective excitation spectrum of a solid and of a nuclear matter, considering that it is, in a good approximation, a collection of strongly interacting electrons and nucleons correspondingly wandering solo in all directions. When discussing nuclear collective motion, one is usually making analogies to molecules and their collective modes [3,11]. The situation in case of molecules is governed by the adiabatic approximation by Born and Oppenheimer which is justified due to different time scales of nuclear and electronic motions. But in nuclei there is no essential difference between the masses of protons and neutrons, and the choice of proper nuclear collective coordinates is a long-standing problem. The same analogy can be traced between the collective modes in nuclear matter and in solids with the similar situation about the masses of their components.

The existence of different time scales in the adiabatic approximation is a special case of the general Bogolubov’s principle of time scale hierarchy, which can be determined not only by the initial large difference of masses which permits separating the system of coordinates into slow (relevant) and fast (irrelevant) ones. The time scale hierarchy may be caused by a strong interaction between particles which reveals by the appearance of large effective masses. The question that appears here, is whether this strong interaction wouldn’t destroy the validity of kinetic equation used for the description of the collective motion. The obtained result for the LS equation for the quasiparticle distribution function makes lawful the following considerations. The time $T$ of the quasiparticle energy formation can be estimated as $\frac{\hbar}{E_F}$, where $E_F$ is a Fermi energy [12]. For a nuclear system this time is of an order $10^{-22}$ sec. and it is the time separation between single-particle and collective motion. Thus, it opens the possibility for a rigorous explanation of the validity of the adiabatic approximation for the description of the collective excitation spectrum in nuclear matter as it is accepted in literature [11]. This problem will be analyzed in a separate paper.

References

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