Inverse Domination Numbers and Disjoint Domination Numbers of Graphs under Some Binary Operations

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Abstract
In this note, we investigate the inverse domination numbers and the disjoint pair domination numbers of graphs resulting from the join, corona and composition of graphs.

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1 Introduction

Throughout this study, $G$ denotes a graph which is simple and undirected. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively. We write $uv$ to denote the edge joining the vertices $u$ and $v$. The order (resp. size) of $G$ refers to the cardinality of $V(G)$ (resp. $E(G)$). In symbols, $|V(G)|$ denotes the order, while $|E(G)|$ denotes the size of $G$. If $E(G) = \emptyset$, $G$ is called an empty graph. If $V(G)$ is a singleton, $G$ is called a trivial graph.

Any graph $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a non-empty $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph $H$ of $G$ for which $|E(H)|$ is the maximum size of a subgraph of $G$ with vertex set $S$.

An edge $e$ of $G$ is said to be incident to vertex $v$ whenever $e = uv$ for some $u \in V(G)$. We write $G - v$ to denote the resulting subgraph of $G$ after removing from $G$ the vertex $v$ and all edges of $G$ incident to $v$. In general, for $S \subseteq V(G)$, the symbol $G - S$ denotes the resulting subgraph of $G$ after removing all vertices $v \in S$ from $G$ and all edges in $G$ incident to $v$. If $u, v \in V(G)$, the symbol $G + uv$ denotes the graph obtained from $G$ by adding to $G$ the edge $uv$.

Let $G$ and $H$ be any graphs. The join of $G$ and $H$ is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The corona $G \circ H$ of $G$ and $H$ is the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$, and then joining the $i^{th}$ vertex of $G$ to every vertex in the $i^{th}$ copy of $H$. We denote by $H^v$ that copy of $H$ whose vertices are adjoined with the vertex $v$ of $G$. In effect, $G \circ H$ is composed of the subgraphs $H^v + v = H^v + \langle \{v\} \rangle$ joined together by the edges of $G$. The composition $G[H]$ of $G$ and $H$ is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only either $uu' \in E(G)$ or $u = u'$ and $vv' \in V(H)$.

Two distinct vertices $u$ and $v$ of $G$ are neighbors in $G$ if $uv \in E(G)$. The closed neighborhood $N_G[v]$ of a vertex $v$ of $G$ is the set consisting of $v$ and every neighbor of $v$ in $G$. Any $S \subseteq V(G)$ is a dominating set in $G$ if $\cup_{v \in S} N_G[v] = V(G)$. The minimum cardinality $\gamma(G)$ of a dominating set in $G$ is the domination number of $G$. Any dominating set in $G$ of cardinality $\gamma(G)$ is referred to as a $\gamma$-set in $G$. A dominating set $S$ in $G$ is a total dominating set if for every $x \in S$ there exists $y \in S$ such that $xy \in E(G)$. The minimum cardinality of a total dominating set in $G$ is the total domination number of $G$, and is denoted by $\gamma_t(G)$. The reader may refer to [3, 9, 11, 13, 14, 15] for the fundamental concepts and recent developments of the domination theory, including its various applications.

A classical result in domination theory due to Ore[3] in 1962 motivated the introduction of the concept of an inverse dominating set. It can be stated as follows:
Theorem 1.1 [3] Let $G$ be a graph with no isolated vertex. If $S \subseteq V(G)$ is a $\gamma$-set in $G$, then $V(G) \setminus S$ is also a dominating set in $G$.

Let $G$ be a graph without isolated vertices. An *inverse dominating set* in $G$ is any dominating set $S$ in $G$ such that $S \subseteq V(G) \setminus D$, where $D$ is a $\gamma$-set in $G$. The minimum cardinality of an inverse dominating set is called the *inverse domination number*, and is denoted by $\gamma'(G)$. Such definition was first introduced by V.R. Kulli and S.C. Sigarkanti [1] in 1991, and studied further in [2, 7, 8]. It may be noted that P.G. Bhat and S.R. Bhat in [2] made mention of its application in an Information Retrieval System. It can be readily verified that $\gamma(G) \leq \gamma'(G)$. T. Tamizh Chelvam, T. Asir and G.S. Grace Prema in [7] studied graphs $G$ where $\gamma(G) = \gamma'(G)$.

For our purposes in this paper, any inverse dominating set $S$ in $G$ with $|S| = \gamma'(G)$ is called a $\gamma'$-set in $G$.

Theorem 1.1 also guarantees that any graph $G$ with no isolated vertices contains two disjoint subsets of $V(G)$ which are both dominating sets in $G$. This is the motivation of the concept of disjoint domination introduced by S.M. Hedetniemi et al. in [10]. Any pair of subsets $S$ and $D$ of $V(G)$ is called *dd-pair* if $S$ and $D$ are disjoint dominating sets in $G$. We define

$$\gamma\gamma(G) = \min\{|S| + |D| : S, D \text{ are dd-pairs in } G\}.$$ 

Any dd-pair $(S, D)$ in $G$ satisfying $|S| + |D| = \gamma\gamma(G)$ is called $\gamma\gamma$-pair in $G$. It is easy to verify that

$$2\gamma(G) \leq \gamma\gamma(G) \leq \gamma'(G) + \gamma(G). \quad (1)$$

For graphs where $\gamma'(G) = \gamma(G)$, $\gamma\gamma(G) = 2\gamma(G)$.

2 Realization Problems

Proposition 2.1 For every pair $(a, b)$ of positive integers with $a \leq b$, there exists a graph $G$ such that $\gamma(G) = a$, $\gamma'(G) = b$ and $\gamma\gamma(G) = a + b$.

Proof: If $a = 1$, then we take $G = K_{1,b}$. Suppose that $a \geq 2$. Let $n = 3a - 2$ and let the path $P_n$ be given by $P_n = [v_1, v_2, \ldots, v_n]$. Form $G$ by adding to $P_n$, $c = b - \left\lfloor \frac{n}{3} \right\rfloor$ pendant edges $u_jv_n, j = 1, 2, \ldots, c$. If $c = 1$, then $\gamma(G) = \gamma(P_{n+1}) = \left\lfloor \frac{n+1}{3} \right\rfloor = a$, while $\gamma'(G) = \gamma'(P_{n+1}) = b$ [1]. Suppose that $c \geq 2$. Since $D = \{v_1, v_4, \ldots, v_n\}$ is a dominating set in $G$,

$$\gamma(G) \leq \left\lfloor \frac{n}{3} \right\rfloor = \left\lfloor \frac{3a - 2}{3} \right\rfloor = a.$$ 

On the other hand, since $\gamma(P_{n+1}) = \left\lfloor \frac{n+1}{3} \right\rfloor \geq \left\lfloor \frac{n}{3} \right\rfloor = a$, $\gamma(G) \geq \gamma(P_{n+1}) \geq a$. Therefore, $\gamma(G) = a$. Consequently, $D$ is a $\gamma$-set in $G$. Note further that,
in particular, the set \( S = \{ v_{n-2}, v_{n-5}, \ldots, v_2 \} \cup \{ u_i : i = 1, 2, \ldots, c \} \) is a dominating in \( G \) and \( S \subseteq V(G) \setminus D \) so that \( \gamma'(G) \leq \left\lceil \frac{n}{3} \right\rceil + c = b \). Now, let \( T \subseteq V(G) \) be a \( \gamma \)-set in \( G \). Since \( \gamma(P_n) = a \) and \( c \geq 2 \), \( u_i \notin T \) for all \( i = 1, 2, \ldots, c \). Consequently, \( v_n \in T \). Let \( D_0 \subseteq V(G) \setminus T \) be an inverse dominating set in \( G \). Since \( v_n \notin D_0 \), \( u_i \in D_0 \) for all \( i \). Similarly, since \( v_1 \notin D_0 \), \( v_2 \in D_0 \). Apparently, the definition of \( D_0 \) implies that \( D_0 = S \). Therefore, \( \gamma'(G) = b \). Finally, let \( (S, D') \) be a \( \gamma \gamma \)-pair in \( G \). Either each \( u_j \in D' \) for each \( j \) or \( u_j \in S \) for each \( j \). If \( u_j \in D' \) for all \( j \), then \( D' = D \), and the conclusion follows.

**Theorem 2.2** For each integer \( n \geq 1 \), there exists a connected graph \( G \) such that \( \gamma'(G) - \gamma(G) = n \) and \( |V(G)| = \gamma'(G) + \gamma(G) \).

**Proof:** Let \( n \geq 1 \), and consider the star graph \( K_{1,n+2} = K_1 + \overline{K_{n+2}} \). Let \( \{ v \} = V(K_1) \) and let \( u \in V(K_{n+2}) \). Obtain the graph \( G \) by adding to \( K_{1,n+2} \) a pendant \( uz \). Then \( \gamma(G) = 2 \), which is determined by the dominating set \( \{ v, z \} \) in \( G \). Since \( S = V(G) \setminus \{ v, z \} \) is a dominating set in \( G \), \( S \) is an inverse dominating set in \( G \) and \( \gamma'(G) \leq |S| = n + 2 \). But since \( N_G[D] \neq V(G) \) for all proper subsets \( D \) of \( S \), \( \gamma'(G) = |S| = n + 2 \). Thus, \( \gamma'(G) - \gamma(G) = n \). \( \blacksquare \)

**Corollary 2.3** The difference \( \gamma'(G) - \gamma(G) \) can be made arbitrarily large.

**Theorem 2.4** For each integer \( n \geq 1 \), there exists a connected graph \( G \) such that \( \gamma(G) + \gamma'(G) - \gamma\gamma(G) = n \).

**Proof:** Consider the graph \( G \) as in Figure 1 obtained by adding to the corona

![Graph G](image)

**Figure 1:** Graph \( G \) with \( \gamma\gamma(G) < \gamma(G) + \gamma'(G) \).

\( K_3 \circ C_4 \) \( n \) vertices \( x_1, x_2, \ldots, x_n \) and the edges \( x_jw, x_ju \) and \( x_jv \) (\( j = 1, 2, \ldots, n \)). The set \( \{ u, v, w \} \) is the unique minimum dominating set in \( G \),
Corollary 2.5 The difference \(\gamma(G) + \gamma'(G) - 2\gamma(G)\) can be made arbitrarily large.

3 Join of graphs

Clearly, \(\gamma'(G + K_1) = \gamma(G)\). In what follows, we consider \(G + H\) with nontrivial graphs \(G\) and \(H\). For any \(u \in V(G)\) and \(v \in V(H)\), the set \(\{u, v\}\) is a dominating set in \(G + H\). Thus, \(\gamma(G + H) \leq 2\).

Lemma 3.1 For nontrivial graphs \(G\) and \(H\), \(\gamma'(G + H) \leq 2\).

Proof: Either \(\gamma(G + H) = 1\) or \(\gamma(G + H) = 2\). Suppose that \(\gamma(G + H) = 1\), and let \(D = \{v\}\) be a dominating set in \(G + H\). Assume \(v \in V(G)\). Take \(u \in V(G) \setminus \{v\}\) and \(w \in V(H)\). Then \(S = \{u, w\} \subseteq V(G + H) \setminus D\) and \(S\) is a dominating set in \(G + H\). Thus \(\gamma'(G + H) \leq |S| = 2\). Suppose that \(\gamma(G + H) = 2\). Pick any \(u \in V(G)\) and \(v \in V(H)\). Then \(D = \{u, v\}\) is a \(\gamma\)-set in \(G + H\). For any \(x \in V(G) \setminus D\) and \(y \in V(H) \setminus D\), the set \(S = \{x, y\}\) is a \(\gamma\)-set in \(G + H\). Thus \(\gamma'(G + H) = |S| = 2\).

Theorem 3.2 Let \(G\) and \(H\) be nontrivial graphs. Then \(\gamma'(G + H) = 2\) if and only if one of the following is true:

(i) \(\gamma(G) \geq 2\) and \(\gamma(H) \geq 2\);

(ii) \(\gamma(H) \geq 2\) and \(G\) has a (unique) vertex that dominates \(V(G)\);

(iii) \(\gamma(G) \geq 2\) and \(H\) has a (unique) vertex that dominates \(V(H)\).

Proof: Suppose that \(\gamma'(G + H) = 2\). Again, either \(\gamma(G + H) = 1\) or \(\gamma(G + H) = 2\). If \(\gamma(G + H) = 2\), then \(\gamma(G) \geq 2\) and \(\gamma(H) \geq 2\). Suppose that \(\gamma(G + H) = 1\). Then \(\gamma(G) = 1\) or \(\gamma(H) = 1\). Assume that \(\gamma(G) = 1\). Then \(G = \{v\} + \bigcup_j G_j\) for some \(v \in V(G)\) and components \(G_j\) of \(G\). Thus,

\[
\gamma'(G + H) = \gamma(H + \bigcup_j G_j) = 2.
\]

Necessarily, \(\gamma(H) \geq 2\) and \(\gamma(\bigcup_j G_j) \geq 2\). This means that \(v\) is a unique vertex of \(G\) that dominates \(V(G)\). Similarly, if \(\gamma(H) = 1\), then \(\gamma(G) \geq 2\) and \(H\) has a unique vertex that dominates \(V(H)\).
To prove the converse, first, consider the case where \( \gamma(G) \geq 2 \) and \( \gamma(H) \geq 2 \). Then \( \gamma(G + H) = 2 \). Now pick \( u \in V(G) \) and \( v \in V(H) \), and choose \( x \in V(G) \setminus \{u\} \) and \( y \in V(H) \setminus \{v\} \). Then \( D = \{u, v\} \) and \( S = \{x, y\} \) are disjoint \( \gamma \)-sets in \( G + H \). Accordingly, \( \gamma'(G + H) = 2 \). Next, suppose that (ii) holds. Let \( D = \{u\} \subseteq V(G) \) be a dominating set in \( G \). Then \( D \) is a dominating set in \( G + H \). Consider \( (G + H) - u = (G - u) + H \).

Since \( u \) is a unique vertex that dominates \( V(G) \), \( \gamma(G - u) \geq 2 \). If \( \gamma(G - u) \geq 2 \) and \( \gamma(H) \geq 2 \), then \( \gamma'(G + H) = \gamma((G - u) + H) = 2 \). Similarly, if (iii) holds, then \( \gamma'(G + H) = 2 \).

**Corollary 3.3** Let \( G \) and \( H \) be nontrivial graphs. Then \( \gamma'(G + H) = 1 \) if and only if one of the following is true:

(i) \( \gamma(G) = 1 \) and \( \gamma(H) = 1 \);

(ii) \( G \) has two distinct vertices each of which dominates \( V(G) \);

(iii) \( H \) has two distinct vertices each of which dominates \( V(H) \).

Corollary 3.3 asserts that for nontrivial graphs \( G \) and \( H \), if \( \gamma'(G + H) = 1 \), then \( \gamma(G) = 1 \) or \( \gamma(H) = 1 \). The converse, however, is not necessarily true. To see this, consider the graph \( K_{1,4} + P_5 \). Note that \( \gamma(K_{1,4}) = 1 \) but \( \gamma'(K_{1,4} + P_5) = 2 \) by Theorem 3.2.

**Corollary 3.4** Let \( G \) be any graph with no isolated vertex. Then \( \gamma'(G) = 1 \) if and only if \( G = K_p \) \( (p \geq 2) \) or \( G = K_2 + H \) for some noncomplete graph \( H \).

**Proof:** First, note that \( \gamma'(K_p) = 1 \) for all \( p \geq 2 \). Thus, we proceed with a noncomplete \( G \). Suppose that \( \gamma'(G) = 1 \). There exist two distinct vertices \( u \) and \( v \) of \( G \) such that \( \{u\} \) and \( \{v\} \) are \( \gamma \)-sets in \( G \). Then \( \{u, v\} = K_2 \) and \( G = K_2 + H \), where \( H = G - \{u, v\} \). The converse follows immediately from Corollary 3.3.

Now we consider pair of disjoint dominating sets in the join of graphs. Clearly, \( \gamma \gamma(G + K_1) = 1 + \gamma(G) = 1 + \gamma'(G + K_1) \) for any graphs \( G \). In particular, \( \gamma \gamma(K_{1,n}) = n + 1 \) for all positive integers \( n \).

**Proposition 3.5** Let \( G \) and \( H \) be nontrivial graphs. Then

\[
2 \leq \gamma \gamma(G + H) \leq 4.
\]

More precisely,
(i) \( \gamma \gamma(G + H) = 2 \) if and only if \( \gamma'(G + H) = 1 \);

(ii) \( \gamma \gamma(G + H) = 3 \) if and only if either \( \gamma(G) \geq 2 \) and \( H \) has a unique vertex that dominates \( V(H) \) or \( \gamma(H) \geq 2 \) and \( G \) has a unique vertex that dominates \( V(G) \);

Proof: From previous discussion,

\[
2 \leq \gamma(G + H) + \gamma(G + H) \leq \gamma(G + H) \leq \gamma(G + H) + \gamma'(G + H) \leq 4.
\]

Statement (i) is clear. Suppose that \( \gamma \gamma(G + H) = 3 \). Then \( \gamma(G + H) = 1 \) and \( \gamma'(G + H) = 2 \). By Theorem 3.2, either \( \gamma(G) \geq 2 \) and \( H \) has a unique vertex that dominates \( V(H) \) or \( \gamma(H) \geq 2 \) and \( G \) has a unique vertex that dominates \( V(G) \). Conversely, by Theorem 3.2, the hypothesis implies that \( \gamma'(G + H) = 2 \) so that \( \gamma \gamma(G + H) \geq 3 \) by Statement (i). The same also implies that \( \gamma(G + H) = 1 \). Therefore, \( \gamma \gamma(G + H) \leq 3 \). This proves Statement (ii).

Corollary 3.6 Let \( G \) and \( H \) be nontrivial graphs. Then \( \gamma \gamma(G + H) = 4 \) if and only if \( \gamma(G) \geq 2 \) and \( \gamma(H) \geq 2 \)

Proof: Suppose that \( \gamma(G) \geq 2 \) and \( \gamma(H) \geq 2 \). Then \( \gamma(G + H) = 2 \). Thus, for any \( dd \)-pair \( S \) and \( D \) in \( G + H \), \( |S| + |D| \geq 4 \). This means that \( \gamma \gamma(G + H) \geq 4 \). Invoking Inequality 2, \( \gamma \gamma(G + H) = 4 \). The converse follows from Proposition 3.5 and Theorem 3.2.

4 Corona of graphs

It is worth noting that for any connected graph \( G \) and for all graphs \( H \), \( V(G) \) is a \( \gamma \)-set in \( G \circ H \). The following theorem is found in [5].

Theorem 4.1 [5] Let \( G \) be a connected graph of order \( m \) and \( H \) any graph of order \( n \). Then \( C \subseteq V(G \circ H) \) is a dominating set in \( G \circ H \) if and only if \( C \cap V(H^v + v) \) is a dominating set in \( H^v + v \) for every \( v \in V(G) \).

Proposition 4.2 For any connected graph \( G \) and for any graph \( H \), \( \gamma'(G \circ H) = |V(G)|\gamma(H) \).

Proof: Suppose that \( \gamma(H) = 1 \). For each \( v \in V(G) \), let \( u^v \in V(H^v) \) such that \( N_{H^v}[u^v] = V(H^v) \). Let \( S \subseteq V(G \circ H) \) be a \( \gamma \)-set in \( G \circ H \). Define

\[
D = \{ v \in V(G) : v \notin S \} \cup \{ u^v : v \in S \cap V(G) \}.
\]

Then \( D \) is a \( \gamma \)-set in \( G \circ H \). Since \( S \cap D = \emptyset \), \( S \) is a \( \gamma' \)-set in \( G \circ H \). Therefore, \( \gamma'(G \circ H) = \gamma(G \circ H) = |V(G)| \).
Suppose that $\gamma(H) > 1$. Let $S \subseteq V(G \circ H)$ be an inverse dominating set in $G \circ H$. For each $v \in V(G)$, let $S_v = S \cap V(H^v + v)$. Since $V(G)$ is the unique $\gamma$-set in $G \circ H$, $S \cap V(G) = \emptyset$. Consequently, $S_v \subseteq V(H^v)$ for all $v \in V(G)$. Moreover, $S_v$ dominates $V(H^v)$. Thus,

$$\gamma'(G \circ H) = |S| = \sum_{v \in V(G)} |S_v| \geq |V(G)| \gamma(H).$$

To get the desired equality, for each $v \in V(G)$, let $S_v \subseteq V(H^v)$ be a $\gamma$-set in $V(H^v)$. Clearly, $S = \bigcup_{v \in V(G)} S_v$ is a dominating set in $G \circ H$. Since $S \cap V(G) = \emptyset$, $S$ is an inverse dominating set in $G \circ H$. Therefore, $\gamma'(G \circ H) \leq |S| = |V(G)| \gamma(H)$.

**Corollary 4.3** For any connected graphs $G$ and for any graph $H$,

$$\gamma \gamma(G \circ H) = |V(G)|(1 + \gamma(H)).$$

**Proof:** Let $S, T \subseteq V(G \circ H)$ and, for each $v \in V(G)$, let $S_v = S \cap V(H^v + v)$ and $T_v = T \cap V(H^v + v)$. By Theorem 4.1, $S$ and $T$ are disjoint dominating sets in $G \circ H$ if and only if $S_v$ and $T_v$ are disjoint dominating sets in $H^v + v$. Moreover, $|S| + |T| = \gamma \gamma(G \circ H)$ if and only if $|S_v| + |T_v| = \gamma \gamma(H^v + v)$ for every $v \in V(G)$. Thus, $\gamma \gamma(G \circ H) = \sum_{v \in V(G)} \gamma \gamma(H^v + v) = |V(G)|(1 + \gamma(H)).$ □

## 5 Composition of graphs

**Theorem 5.1** [6] Let $G$ and $H$ be connected graphs. Then $C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$, is a dominating set in $G[H]$ if and only if either

(i) $S$ is a total dominating set in $G$

(ii) $S$ is a dominating set in $G$ and $T_x$ is a dominating set in $H$ for every $x \in S \setminus N_G(S)$.

**Theorem 5.2** [6] Let $G$ and $H$ be connected graphs with $\gamma(H) \geq 2$. Then $\gamma(G[H]) = \gamma_t(G)$.

**Proposition 5.3** Let $G$ and $H$ be nontrivial connected graphs with $\gamma(H) \geq 2$. Then $\gamma'(G[H]) = \gamma_t(G)$. Consequently, $\gamma \gamma(G[H]) = 2 \gamma_t(G)$.

**Proof:** Let $A \subseteq V(G)$ be a minimum total dominating set in $G$, and let $u, v \in V(H)$, $u \neq v$. By Theorem 5.1, both $S = A \times \{u\}$ and $D = A \times \{v\}$ are (disjoint) dominating sets in $G[H]$. Moreover, $|S| = |D| = |A| = \gamma_t(G) = \gamma(G[H])$, by Theorem 5.2. Thus, $S$ is an inverse dominating set in $G[H]$ so that

$$\gamma_t(G) = \gamma(G[H]) \leq \gamma'(G[H]) \leq |S| = \gamma_t(G).$$

This proves the proposition. □
Lemma 5.4  Let $G$ and $H$ be nontrivial connected graphs such that $V(H)$ is dominated by a vertex $v \in V(H)$. If $A \subseteq V(G)$ is an inverse dominating set in $G$, then $A \times \{v\}$ is an inverse dominating set in $G[H]$.

**Proof:** Let $A, B \subseteq V(G)$ be dominating sets in $G$ such that $A \cap B = \emptyset$ and $|B| = \gamma(G)$. By Theorem 5.1, $A \times \{v\}$ and $B \times \{v\}$ are dominating sets in $G[H]$. Now, $\gamma(G[H]) \leq |B \times \{v\}| = |B| = \gamma(G) \leq \gamma(G[H])$ so that $B \times \{v\}$ is a $\gamma$-set in $G[H]$. Since $(A \times \{v\}) \cap (B \times \{v\}) = \emptyset$, $A \times \{v\}$ is an inverse dominating set in $G[H]$. \hfill \Box

For convenience, we write $S^o = S \setminus N_G(S)$ for any $S \subseteq V(G)$.

**Theorem 5.5** Let $G$ and $H$ be nontrivial connected graphs with $\gamma(H) = 1$. Then

$$\gamma(G) \leq \gamma'(G[H]) \leq \gamma'(G).$$

(3)

More precisely,

(i) if $H$ has (at least) two distinct vertices each of which dominates $V(H)$, then $\gamma'(G[H]) = \gamma(G)$; and

(ii) if $H$ has a unique vertex that dominates $V(H)$, then

$$\gamma'(G[H]) = \min\{(|A| + |A^o \cap B|)(\gamma'(H) - 1) : A, B \in \Gamma(G)$$

$$\text{with } |B| = \gamma(G)\},$$

where $\Gamma(G)$ is the family of all dominating sets in $G$.

**Proof:** Inequality 3 follows immediately from Lemma 5.4. Suppose that $H$ has two distinct vertices $u$ and $v$ such that $N_H[u] = V(H) = N_H[v]$. Let $A \subseteq V(G)$ be a $\gamma$-set in $G$. By Theorem 5.1, $S = A \times \{u\}$ and $D = A \times \{v\}$ are $\gamma$-sets in $G[H]$. Since $S \cap D = \emptyset$, $S$ is a $\gamma'$-set in $G[H]$. Hence, $\gamma'(G[H]) = |S| = |A| = \gamma(G)$.

Suppose that $H$ has a unique vertex $v$ that dominates $V(H)$. Let $\Gamma = \Gamma(G)$ denote the family of all dominating sets in $G$, and let

$$\alpha = \min\{(|A| + |A^o \cap B|)(\gamma'(H) - 1) : A, B \in \Gamma \text{ with } |B| = \gamma(G)\}.$$

Let $A, B \in \Gamma(G)$ with $|B| = \gamma(G)$, and let $v \in V(H)$ such that $N_H[v] = V(H)$. Choose $w \in V(H) \setminus \{v\}$ and a $\gamma'$-set $C \subseteq V(H)$ in $H$. It is worth noting that $v \notin C$. Define $D = B \times \{v\}$ and

$$S = (\cup_{u \in A \setminus B} \{(u, v)\}) \cup (\cup_{u \in A \setminus A^o \cap B} \{(u, w)\}) \cup (\cup_{u \in A \setminus B} \{u\} \times C).$$
By Theorem 5.1 and the fact that \(|D| = |B| = \gamma(G)|, D is a \(\gamma\)-set in \(G[H]\).

Let \(u \in A^0\). Then \(T_u = \{x \in V(H) : (u, x) \in S\}\) is either \(C\) or \(\{v\}\). In any case, \(T_u\) is a dominating set in \(H\). By Theorem 5.1, \(S\) is a dominating set in \(G[H]\). Since \(S \cap D = \emptyset\), \(S\) is an inverse dominating set in \(G[H]\). Thus,
\[
\gamma'(G[H]) \leq |S| = |A| + |A^0 \cap B|((\gamma'(H) - 1))
\]

Since \(A\) and \(B\) are arbitrary, \(\gamma'(G[H]) \leq \alpha\).

Let \((S, D)\) be a \(dd\)-pair in \(G[H]\) such that \(|D| = \gamma(G[H])\) and \(|S| = \gamma'(G[H])\). By Theorem 5.1, \(S = \bigcup_{u \in A}(\{u\} \times T_u)\) and \(D = \bigcup_{u \in B}(\{u\} \times T_u)\) for some dominating sets \(A\) and \(B\) in \(G\). Since \(\gamma(H) = 1\), Theorem 5.1 implies that \(|B| = |D| = \gamma(G)\) and \(|T_u| = 1\) for all \(u \in B\). Since \(S\) is a \(\gamma\)-set, \(|T_u| = 1\) for all \(u \in A \setminus B\), in which case, we may assume that \(T_u = \{v\} \subseteq V(H)\) where \(N_H[v] = V(H)\). Since \(S \cap D = \emptyset\), for all \(u \in A^0 \cap B\), if \((u, w) \in D\), then \((u, w) \notin S\). Moreover, in view of Theorem 5.1(ii), for each such \(u\), \(T_u = \{x \in V(H) : (u, x) \in S\}\) is a \(\gamma\)-set in \(H\). Thus,
\[
|S| = |\bigcup_{u \in A \setminus B}(\{u\} \times T_u)| + |\bigcup_{u \in (A \cap B) \cap B}(\{u\} \times T_u)| + |\bigcup_{u \in A^0 \cap B}(\{u\} \times T_u)|
\]
\[
\geq |A \setminus (A^0 \cap B)| + |A^0 \cap B|\gamma'(H)
\]
\[
= |A| + |A^0 \cap B|((\gamma'(H) - 1))
\]
so that \(\gamma'(G[H]) \geq \alpha\).

**Corollary 5.6** Let \(G\) and \(H\) be nontrivial connected graphs. If \(H\) has a unique vertex that dominates \(V(H)\), then \(\gamma'(G[H]) = \gamma'(G)\) if and only if \(G\) has an inverse dominating set \(A_0\) such that \(|A_0| \leq |A| + |A^0 \cap B|((\gamma'(H) - 1))\) for all \(dd\)-pairs \(A\) and \(B\) in \(G\) with \(|B| = \gamma(G)\).

The inequalities in Inequality 3 can be both strict. Consider, for example, the composition \(G[P_3]\), where \(G\) is the graph in Figure 2. Verify that \(\gamma(G) = 2\),

![Figure 2: Graph G where \(\gamma(G) < \gamma'(G[P_3]) < \gamma'(G)\)](image)

\(\gamma'(G[P_3]) = 3\) and \(\gamma'(G) = 5\). The set \(B = \{u, w\}\) is the unique \(\gamma\)-set in \(G\). Consider \(A = \{u, v, w\}\), which is a total dominating set in \(G\) so that \(A^0 = \emptyset\). Applying Theorem 5.5(ii), \(\gamma'(G[P_3]) = |A|\).

Inequality 3 implies that for connected graphs \(G\) and \(H\) with \(\gamma(H) = 1\),
\[
\gamma\gamma(G[H]) \leq \gamma(G) + \gamma'(G).
\]

The next result is an improvement of inequality 4.
Theorem 5.7 Let $G$ and $H$ be nontrivial connected graphs with $\gamma(H) = 1$. Then

$$2\gamma(G) \leq \gamma\gamma(G[H]) \leq \gamma(G).$$

More precisely,

(i) if $H$ has (at least) two distinct vertices each of which dominates $V(H)$, then $\gamma\gamma(G[H]) = 2\gamma(G)$; and

(ii) if $H$ has a unique vertex that dominates $V(H)$, then

$$\gamma\gamma(G[H]) = \min\{|A| + |B| + |A^0 \cap B^0|((\gamma'(H) - 1)) : A, B \in \Gamma(G)\},$$

where $\Gamma(G)$ is the family of all dominating sets in $G$.

Proof: There exists $v \in V(H)$ such that $N_H[v] = V(H)$. Let $(A, B)$ be a $\gamma\gamma$-pair in $G$. Then $(A \times \{v\}, B \times \{v\})$ is a $dd$-pair in $G[H]$. Thus, $\gamma\gamma(G[H]) \leq |A \times \{v\}| + |B \times \{v\}| = |A| + |B| = \gamma(G)$.

If $H$ has two distinct vertices that both dominate $V(H)$, then Theorem 5.5(i) implies

$$2\gamma(G) \leq \gamma\gamma(G[H]) \leq \gamma(G[H]) + \gamma'(G[H]) = 2\gamma(G).$$

Suppose that $H$ has a unique vertex $v$ that dominates $V(H)$. Let

$$\alpha = \min\{|A| + |B| + |A^0 \cap B^0|((\gamma'(H) - 1)) : A, B \in \Gamma(G)\}.$$ 

Let $w \in V(H) \setminus \{v\}$, let $A, B \in \Gamma(G)$ and $(X, Y)$ a $dd$-pair in $H$. Define

$$S = (\bigcup_{u \in (A \setminus A^0) \cap B^0} \{(u, w)\}) \cup (\bigcup_{u \in A \cap B^0} \{(u, v)\}) \cup (\bigcup_{u \in A^0 \cap B^0} \{(u) \times X\}),$$

and $D = \bigcup_{u \in B} \{(u) \times T_u\}$ such that

(a) for each $u \in A^0 \cap B^0$, $T_u = Y$;

(b) for each $u \in (B \setminus A) \cup ((A \setminus A^0) \cap B^0)$, $T_u = \{v\}$; and

(c) for each $u \in [(B \setminus B^0) \cap A^0] \cup [(A \setminus A^0) \cap (B \setminus B^0)]$, $T_u = \{w\}$.

By Theorem 5.1, $S$ and $D$ are dominating sets in $G[H]$. Moreover, $S \cap D = \emptyset$. Thus,

$$\gamma\gamma(G[H]) \leq |S| + |T| = |A| + |B| + |A^0 \cap B^0|(|X| + |Y| - 2).$$

Since $X$ and $Y$ are arbitrary,

$$\gamma\gamma(G[H]) \leq |A| + |B| + |A^0 \cap B^0|((\gamma'(H) - 2)) = |A| + |B| + |A^0 \cap B^0|((\gamma'(H) - 1)).$$
Since $A$ and $B$ are arbitrary, $\gamma\gamma(G[H]) \leq \alpha$.

To prove the converse, let $(S,D)$ be a $\gamma\gamma$-pair in $G[H]$. There exist dominating sets $A$ and $B$ in $G$ such that $S = \cup_{u \in A}(\{u\} \times T_u)$ and $D = \cup_{u \in B}(\{u\} \times T_u)$. Further, if $A$ (resp. $B$) is not a total dominating set in $G$, then for each $u \in A^\circ$ (resp $B^\circ$), $T_u$ is a dominating set in $H$. In view of Theorem 5.1, since $(S,D)$ is a $\gamma\gamma$-pair in $G[H]$, we have for each $u \in A^\circ \cap B^\circ$, $(u,y) \in S$ and $(y \in V(H) : (u,y) \in D)$ constitute a $\gamma\gamma$-pair in $H$. Thus,

$$\gamma\gamma(G[H]) = |S| + |D| \geq |A| + |B| + |A^\circ \cap B^\circ| (\gamma\gamma(H) - 2) \geq \alpha.$$ 

This proves Statement (ii).

\textbf{Corollary 5.8} Let $G$ and $H$ be nontrivial connected graphs. If $H$ has a unique vertex that dominates $V(H)$, then $\gamma\gamma(G[H]) = \gamma\gamma(G)$ if and only if $G$ has a $\gamma\gamma$-pair $(A_0, B_0)$ such that $|A_0| + |B_0| \leq |A| + |B| + |A^\circ \cap B^\circ| (\gamma'(H) - 1)$ for all dominating sets $A$ and $B$ in $G$.

\textbf{Example 5.9} (1) For all integers $n, m \geq 3$,

$$\gamma'(K_{1,n}[K_{1,m}]) = 2 \text{ and } \gamma\gamma(K_{1,n}[K_{1,m}]) = 3.$$ 

(2) For noncomplete connected graphs $G$ and integers $p \geq 2$,

$$\gamma'(G[K_p]) = \gamma(G) \text{ and } \gamma\gamma(G[K_p]) = 2\gamma(G).$$

(3) For noncomplete graphs $G$ and integers $p \geq 2$,

$$\gamma'(K_p[G]) = \begin{cases} 1, & \text{if } \gamma(G) = 1, \\ 2, & \text{if } \gamma(G) \geq 2 \end{cases}$$ 

and

$$\gamma\gamma(K_p[G]) = \begin{cases} 2, & \text{if } \gamma(G) = 1, \\ 4, & \text{if } \gamma(G) \geq 2. \end{cases}$$

\textbf{References}


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