A Fixed Point Theorem for Generalized Weakly α-Contractive Mappings in Cone Metric Spaces

Ji-Young Lee, Seong-Hoon Cho* and Gwang-Yeon Lee

Department of Mathematics
Hanseo University, Seosan
Chungnam, 356-706, South Korea
*Corresponding author

Copyright © 2014 Ji-Young Lee, Seong-Hoon Cho and Gwang-Yeon Lee. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we introduce the concept of generalized weakly α-contractive mappings, and we establish a new fixed point theorem for such mappings. We give an example to illustrate main result. Also, we give an application to integral equations.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Fixed point, Contractive mapping, Generalized weak α-contractive mapping, Metric space, Cone metric space

1 Introduction and preliminaries

Alber et al. [1] introduced the notion of weakly contractive mappings in Hilbert spaces and proved that any weakly contractive mapping defined on complete Hilbert spaces has a unique fixed point.

Since then, many authors ([5, 8, 15, 17, 18, 19, 20, 21, 24, 26, 28]) investigated fixed point results for weakly contractive type mappings.

Huang and Zhang [22] introduced cone metric spaces which are generalizations of metric spaces, and they extended the Banach’s contraction principle to such spaces.
Then, many authors ( [6, 7, 9, 10, 11, 12, 13, 14, 23, 27] and reference therein ) studied fixed point theorems in cone metric spaces.

Especially, the authors of [14] extend weak contraction principle to cone metric spaces.

Altun et al. [3] introduced the notion of partially ordered cone metric spaces and they extended Banach’s fixed point result to ordered cone metric spaces. And then, Altun et al. [4] obtained Ciric type fixed point results. The authors of [16] proved fixed point results for weakly contractive type mappings in ordered cone metric spaces.

Recently, the authors of [25] introduced the notion of \( \alpha - \phi \)-contractive mapping in metric spaces, and they have some fixed point theorems for such mappings.

Very recently, the author of [8] introduced a concept of weakly \( \alpha \)-contractive mappings and gave a fixed point theorem for such mappings. He proved the following theorem.

**Theorem 1.1.** [8] Let \( (X,d) \) be a complete metric space. Suppose that \( T : X \to X \) is a weakly \( \alpha \)-contractive mapping, i.e.

\[
\alpha(x,y)d(Tx,Ty) \leq d(x,y) - \psi(d(x,y))
\]

for all \( x,y \in X \), where \( \alpha : X \times X \to [0,\infty) \) is a function and \( \psi : [0,\infty) \to [0,\infty) \) is a nondecreasing function such that \( \psi \) is positive on \((0,\infty), \psi(0) = 0 \) and \( \lim_{t \to \infty} \psi(t) = \infty \).

Assume that the following conditions are satisfied:

1. for each \( x \in X \), \( \alpha(x,y) \geq 1 \) and \( \alpha(y,z) \geq 1 \) implies \( \alpha(x,z) \geq 1 \);
2. \( T \) is \( \alpha \)-admissible, i.e. for all \( x,y \in X \), \( \alpha(x,y) \geq 1 \) implies \( \alpha(Tx,Ty) \geq 1 \);
3. there exists \( x_0 \in X \) such that \( \alpha(x_0,Tx_0) \geq 1 \);
4. either \( T \) is continuous or

\[
\lim_{n \to \infty} \inf \alpha(T^n x_0, x) > 0
\]

for any cluster point \( x \) of \( \{T^n x_0\} \).

Then \( T \) has a fixed point in \( X \). Further if, for any \( x,y \in X \), there exists \( z \in X \) such that \( \alpha(x,z) \geq 1 \) and \( \alpha(y,z) \geq 1 \), then \( T \) has a unique fixed point in \( X \).
In the paper, we introduce a notion of generalized weakly $\alpha$-contractive mappings in cone metric spaces and establish a new fixed point theorem for such mappings, which is generalizations of the results in [14, 16] and extensions of the results in [5, 8, 20] to the case of cone metric spaces.

Finally, we give an application of our result for generalized weakly $\alpha$-contractive mappings to integal equations.

Consistent with Huang and Zhang [22], the following definitions will be needed in the sequel.

Let $E$ be a real Banach space. A subset $P$ of $E$ is called a cone if the following conditions are satisfied:

1. $P$ is nonempty closed and $P \neq \{0\}$;
2. $ax + by \in P$, whenever $x, y \in P$ and $a, b \in \mathbb{R}(a, b \geq 0)$;
3. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$.

For $x, y \in P$, $x \ll y$ stand for $y - x \in \text{int}(P)$, where $\text{int}(P)$ is the interior of $P$. A cone $P$ is called normal if there exists a number $K \geq 1$ such that for all $x, y \in E$, $\|x\| \leq K\|y\|$ whenever $0 \leq x \leq y$.

A cone $P$ is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{u_n\}$ is a sequence such that for some $z \in E$

$$u_1 \leq u_2 \leq \cdots \leq z,$$

then there exists $u \in E$ such that

$$\lim_{n \to \infty} \|u_n - u\| = 0.$$

Equivalently, a cone $P$ is regular if and only if every decreasing sequence which is bounded from below is convergent.

It is well known that every regular cone is normal.

From now on, we assume that $E$ is a real Banach space, $P$ is a cone in $E$ with $\text{int}(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to $P$.

Let $X$ be a nonempty set.

Then, a mapping $d : X \times X \to E$ is called cone metric on $X$ if the following conditions are satisfied:

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Example 1.1. [22] Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$. Then $P$ is a regular cone. Let $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|,$
, where \( k \) is a positive constant. Then \((X, d)\) is a cone metric space.

**Example 1.2.** [23] Let \( E = C^1_0([0, 1]) \) with the norm \( \|f\| = \|f\|_{\infty} + \|f'\|_{\infty} \) and \( P = \{ f \in E : f \geq 0 \} \). Then \( P \) is a non-normal cone.

Let \( X = [0, \infty) \). Define \( d : X \times X \to E \) by \( d(x, y) = |x - y| f \), where \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(t) = e^t \). Then \((X, d)\) is a cone metric space.

**Example 1.3.** [7] Let \( X = L^1[0, 1] \), \( E = C[0, 1] \) and \( P = \{ f \in E : f \geq 0 \} \). Then \( P \) is a normal cone with normal constant \( K = 1 \). Define \( d : X \times X \to E \) by \( d(f, g)(t) = \int_0^t |f(x) - g(x)|dx \), where \( 0 \leq t \leq 1 \). Then \((X, d)\) is a complete cone metric space.

A sequence \( \{x_n\} \) in a cone metric space \((X, d)\) converges to a point \( x \in X \) (denoted by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \)) if for any \( c \in int(P) \), there exists \( N \) such that for all \( n > N \), \( d(x_n, x) \ll c \). A sequence \( \{x_n\} \) in a cone metric space \((X, d)\) is Cauchy if for any \( c \in int(P) \), there exists \( N \) such that for all \( n, m > N \), \( d(x_n, x_m) \ll c \). A cone metric space \((X, d)\) is called complete if every Cauchy sequence is convergent.

Note that if \( \lim_{n \to \infty} d(x_n, x) = 0 \), then \( \lim_{n \to \infty} x_n = x \). The converse is true if \( P \) is a normal cone. Also, if \( P \) is a normal cone, then \( \{x_n\} \) is a Cauchy sequence in \( X \) if and only if \( \lim_{m,n \to \infty} d(x_n, x_m) = 0 \).

**Lemma 1.1.** [16] Let \((X, d)\) be a cone metric space. Suppose that a function \( \phi : int(P) \cup \{0\} \to int(P) \cup \{0\} \) satisfies the following:

1. \( \phi(t) = 0 \) if and only if \( t = 0 \);
2. \( \phi(t) \ll t \) for all \( t \in int P \);
3. either \( \phi(t) \leq d(x, y) \) or \( d(x, y) \ll \phi(t) \) for all \( t \in int(P) \cup \{0\} \) and \( x, y \in X \).

If a sequence \( \{x_n\} \subset X \) is not Cauchy, then we have:

1. there exists \( c \in E \) with \( 0 \ll c \) such that, for all \( k > 0 \), there exist \( m(k) > n(k) > k \) with \( \phi(c) \leq d(x_{m(k)}, x_{n(k)}) \) and \( d(x_{m(k)-1}, x_{n(k)}) \ll \phi(c) \), where \( m(k) \) is the smallest such positive integer.
2. \( \lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k \to \infty} d(x_{m(k)+1}, x_{n(k)+1}) = \lim_{k \to \infty} d(x_{m(k)+1}, x_{n(k)}) = \phi(c) \).
2 Fixed point theorems

Let $(X,d)$ be a cone metric space.

We denote by $\Psi$ the family of all nondecreasing continuous functions $\psi : P \to P$ such that

$$\psi(t) = 0 \text{ if and only if } t = 0.$$ 

Also, we denoted by $\Phi$ the class of all continuous functions $\phi : \text{int}(P) \cup \{0\} \to \text{int}(P) \cup \{0\}$ satisfy the following conditions:

1. $\phi(t) = 0$ if and only if $t = 0$;
2. $\phi(t) \ll t$ for all $t \in \text{int}P$;
3. either $\phi(t) \leq d(x,y)$ or $d(x,y) \ll \phi(t)$ for all $t \in \text{int}(P) \cup \{0\}$ and $x,y \in X$.

From now on, let $\alpha : X \times X \to [0, \infty)$ be a function, $\psi \in \Psi$ and $\phi \in \Phi$ such that $\psi(t) - \phi(s) \in P$ for all $t,s \in \text{int}(P) \cup \{0\}$.

A mapping $T : X \to X$ is called \textit{generalized weakly $\alpha$-contractive} if

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq \psi(m(x,y)) - \phi(d(x,y)) \quad (2.1)$$

for all $x,y \in X$, where $m(x,y) = pd(x,y) + qd(x,Tx) + rd(y,Ty) + s[d(x,Ty) + d(y,Tx)]$ with $p,q,r,s \geq 0$ and $p + q + r + 2s \leq 1$.

A mapping $S : X \to X$ is called $\alpha$-admissible [25] if, for all $x,y \in X$, $\alpha(x,y) \geq 1$ implies $\alpha(Sx,Sy) \geq 1$.

**Theorem 2.1.** Let $(X,d)$ be a complete cone metric space with regular cone $P$ such that $d(x,y) \in \text{int}(P)$ for $x,y \in X$ with $x \neq y$. Suppose that a generalized weak $\alpha$-contraction $T : X \to X$ satisfies the following:

1. for each $x,y,z \in X$, $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1$ implies $\alpha(x,z) \geq 1$;
2. $T$ is $\alpha$-admissible;
3. there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$;
4. either $T$ is continuous or if $\{x_n\} \subset X$ is a sequence with $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x$ is a cluster point of $\{x_n\}$ then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \geq 1$. \quad (2.2)

Then $T$ has a fixed point $x_* \in X$ and $T$ is a Picard operator, that is, $\{T^n x_1\}$ converges to $x_* \in X$. 
Proof. Let $x_1 \in X$ be such that $\alpha(x_1, T x_1) \geq 1$. Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = T x_n$ for all $n \in \mathbb{N}$.

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n$ is a fixed point of $T$, and the proof is finished.

Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since $\alpha(x_1, x_2) = \alpha(x_1, T x_1) \geq 1$, from (2) we have $\alpha(x_2, x_3) = \alpha(T x_1, T x_2) \geq 1$.

Again, from (2) we have $\alpha(x_3, x_4) = \alpha(T x_2, T x_3) \geq 1$.

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1$$  \hspace{1cm} (2.3)

for all $n \in \mathbb{N}$.

Applying (2.1) with $x = x_n, y = x_{n+1}$, and using (2.3) we have

$$\psi(d(x_{n+1}, x_{n+2}))$$

$$= \psi(d(T x_n, T x_{n+1}))$$

$$\leq \alpha(x_n, x_{n+1}) \psi(d(T x_n, T x_{n+1}))$$

$$\leq \psi(m(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1}))$$  \hspace{1cm} (2.4)

for all $n \in \mathbb{N}$, where $m(x_n, x_{n+1}) = pd(x_n, x_{n+1}) + qd(x_n, x_{n+1}) + rd(x_{n+1}, x_{n+2}) + s[d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})]$.

Then,

$$m(x_n, x_{n+1})$$

$$= pd(x_n, x_{n+1}) + qd(x_n, x_{n+1}) + rd(x_{n+1}, x_{n+2}) + sd(x_{n+1}, x_{n+1}) + sd(x_{n+1}, x_{n+2})$$

$$\leq pd(x_n, x_{n+1}) + qd(x_n, x_{n+1}) + rd(x_{n+1}, x_{n+2}) + sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2})$$.

Let $M(x_n, x_{n+1}) = pd(x_n, x_{n+1}) + qd(x_n, x_{n+1}) + rd(x_{n+1}, x_{n+2}) + sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2})$.

Then, $m(x_n, x_{n+1}) \leq M(x_n, x_{n+1})$. Since $\psi$ is nondecreasing, form (2.4) we obtain

$$\psi(d(x_{n+1}, x_{n+2}))$$

$$\leq \psi(M(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1}))$$

$$< \psi(M(x_n, x_{n+1})),$$  \hspace{1cm} (2.5)

and so we have

$$d(x_{n+1}, x_{n+2})$$

$$< M(x_n, x_{n+1})$$

$$= pd(x_n, x_{n+1}) + qd(x_n, x_{n+1}) + rd(x_{n+1}, x_{n+2}) + sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2})$$.

which implies that, for all $n \in \mathbb{N}$, $d(x_{n+1}, x_{n+2}) < \frac{p+q+s}{1-r-s}d(x_n, x_{n+1})$. 

Because \( \frac{p+t+s}{r-s} \leq 1 \), \( d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \). It follows that the sequence \( \{d(x_n, x_{n+1})\} \) is nonincreasing. Since cone \( P \) is regular and \( 0 \leq d(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \), there exists \( l \in P \) such that \( \lim_{n \to \infty} d(x_n, x_{n+1}) = l \).

Letting \( n \to \infty \) in the inequality (2.5), and using continuity of \( \psi \) and \( \phi \), we have
\[
\psi(l) \leq \psi((p + q + r + 2s)l) - \phi(l) \leq \psi(l) - \phi(l)
\]
which implies \( \phi(l) = 0 \), and so \( l = 0 \). Thus,
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{2.6}
\]

We now show that \( \{x_n\} \) is a Cauchy sequence.

Suppose that \( \{x_n\} \) is not a Cauchy sequence.

By Lemma 1.1 (a), there exists \( c \in E \) with \( 0 \ll c \) such that, for all \( k > 0 \), there exist \( m(k) \) and \( n(k) > k \) with \( \phi(c) \leq d(x_{m(k)}, x_{n(k)}) \) and \( d(x_{m(k)-1}, x_{n(k)}) \ll \phi(c) \), where \( m(k) \) is the smallest such positive integer.

From (1) and (2.3) we obtain \( \alpha(x_{n(k)}, x_{m(k)}) \geq 1 \) for all \( k \in \mathbb{N} \).

Applying (2.1) with \( x = x_{n(k)} \) and \( y = x_{m(k)} \) we have
\[
\psi(d(x_{n(k)}+1, x_{m(k)+1})) = \psi(d(Tx_{n(k)}, Tx_{m(k)})) \\ \leq \alpha(x_{n(k)}, x_{m(k)}) \psi(d(Tx_{n(k)}, Tx_{m(k)})) \\ \leq \psi(m(x_{n(k)}, x_{m(k)})) - \phi(d(x_{n(k)}, x_{m(k)})). \tag{2.7}
\]

Applying Lemma 1.1 (b), we obtain
\[
\lim_{k \to \infty} m(x_{n(k)}, x_{m(k)}) = \lim_{k \to \infty} [pd(x_{n(k)}, x_{m(k)}) + qd(x_{n(k)}, x_{n(k)+1}) \\ + rd(x_{m(k)+1}, x_{m(k)}) + s[d(x_{n(k)}, x_{m(k)+1}) + d(x_{n(k)+1}, x_{m(k)})]] \\ = (p + 2s)\phi(c). \tag{2.8}
\]

Letting \( k \to \infty \) in (2.7), and applying Lemma 1.1 (b) with (2.8), we have
\[
\psi(\phi(c)) \leq \psi((p + 2s)\phi(c)) - \phi(\phi(c)) \leq \psi(\phi(c)) - \phi(\phi(c))
\]
which is a contradiction.

Thus, \( \{x_n\} \) is a Cauchy sequence.

It follows from the completeness of \( X \) that there exists
\[
x_* = \lim_{n \to \infty} x_n \in X. \tag{2.9}
\]
If $T$ is continuous, then $\lim_{n \to \infty} x_n = Tx$, and so $x_\ast = Tx_\ast$.

Assume that there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$ and any cluster point $x$ of $\{x_n\}$.

Then, we have

$$
\psi(d(x_{n(k)+1}, Tx)) = \psi(d(Tx_{n(k)}, Tx)) \\
\leq \alpha(x_{n(k)}, x) \psi(d(Tx_{n(k)}, Tx)) \\
\leq \psi(m(x_{n(k)}, x)) - \phi(d(x_{n(k)}, x))
$$

for all $k \in \mathbb{N}$.

We have $m(x_{n(k)}, x) = pd(x_{n(k)}, x) + rd(x_{n(k)}, x_{n(k)+1}) + s[\psi(x_{n(k)}, Tx) + d(x_{n(k)+1}, x)]$, and so $\lim_{k \to \infty} m(x_{n(k)}, x) = (r+s)d(x, Tx)$.

Letting $k \to \infty$ in the inequality (2.10), and using continuity of $\psi$ and $\phi$, we have $\psi(d(x, Tx)) \leq \psi((r+s)d(x, Tx)) - \phi(0) = \psi((r+s)d(x, Tx))$.

Since $p + q + r + 2s \leq 1$, $r + s < 1$. Because $\psi$ is nondecreasing,

$$d(x, Tx) \leq (r+s)d(x, Tx)$$

which implies $d(x, Tx) = 0$, and hence $x_\ast = Tx_\ast$. \hfill $\square$

**Example 2.1.** Let $E = \mathbb{R}^2$, $P = \{(s,t) \in E : s, t \geq 0\}$, $X = [0, \infty)$, and let $d : X \times X \to E$ be defined by $d(x, y) = (|x-y|, |x-y|)$. Then $(X, d)$ is a complete cone metric space, $P$ is a regular cone and $d(x, y) \in \text{int}(P)$ for all $x \neq y$.

We define a mapping $T : X \to X$ by

$$
Tx = \begin{cases} 
\frac{1}{2}x & (0 \leq x \leq 1), \\
2x & (x > 1).
\end{cases}
$$

Let $\psi(t) = t$ for all $t \in P$, and $\phi(t) = \frac{1}{4}t$ for all $t \in \text{int}(P) \cup \{0\}$, and let $p = \frac{1}{2}, q = r = s = \frac{1}{5}$.

Then, $T$ do not satisfy (2.13). In fact, $\psi(d(T1, T2)) = (\frac{15}{4}, \frac{15}{4}) > m(2, 1) > m(2, 1) - \phi(d(2, 1))$.

We define a function $\alpha : X \times X \to [0, \infty)$ by

$$
\alpha(x, y) = \begin{cases} 
1 & (0 \leq x, y \leq 1), \\
0 & \text{otherwise}.
\end{cases}
$$

Obviously, condition (1) of Theorem 2.1 is satisfied. Condition (3) of Theorem 2.1 is satisfied with $x_1 = 1$. Condition (4) of Theorem 2.1 is satisfied with $x_n = \frac{1}{n}$. It is easy to see that $T$ is a generalized weakly $\alpha$-contractive mapping.
Let \( x, y \in X \) such that \( \alpha(x, y) \geq 1 \).

Then \( x, y \in [0, 1] \), and so \( Tx \in [0, 1], Ty \in [0, 1] \) and \( \alpha(Tx, Ty) = 1 \). Hence \( T \) is \( \alpha \)-admissible. Thus, all hypothesis of Theorem 2.1 are satisfied, and \( T \) has a fixed point \( x_* = 0 \).

**Corollary 2.2.** Let \( (X, d) \) be a complete cone metric space with regular cone \( P \) such that \( d(x, y) \in \text{int}(P) \) for \( x, y \in X \) with \( x \neq y \). Suppose that a generalized weak \( \alpha \)-contraction \( T : X \to X \) satisfies conditions (1), (2) and (3) in Theorem 2.1.

Assume that either \( T \) is continuous or if \( \{x_n\} \subset X \) is a sequence with \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \) and \( x \) is a cluster point of \( \{x_n\} \) then

\[
\liminf_{n \to \infty} \alpha(x_n, x) \geq 1. \quad (2.11)
\]

Then \( T \) has a fixed point in \( X \).

**Proof.** Note that if \( \lim_{n \to \infty} \sup \alpha(x_n, x) \geq 1 \) then there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n(k)}, x) \geq 1 \) for all \( k \in \mathbb{N} \).

Thus, (2.11) implies (2.2), and hence we have the desired result. \( \square \)

**Corollary 2.3.** Let \( (X, d) \) be a complete cone metric space with regular cone \( P \) such that \( d(x, y) \in \text{int}(P) \) for \( x, y \in X \) with \( x \neq y \). Suppose that a generalized weak \( \alpha \)-contraction \( T : X \to X \) satisfies conditions (1), (2) and (3) in Theorem 2.1.

Assume that either \( T \) is continuous or if \( \{x_n\} \subset X \) is a sequence with \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} x_n = x \), then \( \alpha(x_n, x) \geq 1 \) for all \( n \in \mathbb{N} \).

Then \( T \) has a fixed point in \( X \).

From Theorem 2.1 we have the following corollary.

**Corollary 2.4.** Let \( (X, d) \) be a complete cone metric space with regular cone \( P \) such that \( d(x, y) \in \text{int}(P) \) for \( x, y \in X \) with \( x \neq y \). Let \( T : X \to X \) be a mapping such that

\[
\alpha(x, y)\psi(d(Tx, Ty)) \leq \psi(n(x, y)) - \phi(d(x, y)) \quad (2.12)
\]

for all \( x, y \in X \), where \( n(x, y) = pd(x, y) + q[d(x, Tx) + d(y, Ty)] + s[d(x, Ty) + d(y, Tx)] \) with \( p, q, s \geq 0 \) and \( p + 2q + 2s \leq 1 \).

Suppose that conditions (1), (2), (3) and (4) in Theorem 2.1 are satisfied.

Then \( T \) has a fixed point in \( X \).

**Remark 2.1.** In Corollary 2.2 and 2.3, if we replace a generalized weak \( \alpha \)-contraction by a mapping \( T : X \to X \) satisfying (2.12), then \( T \) has a fixed point.
**Corollary 2.5.** Let \((X, d)\) be a complete cone metric space with regular cone \(P\) such that \(d(x, y) \in \text{int}(P)\) for \(x, y \in X\) with \(x \neq y\). Suppose that a mapping \(T : X \rightarrow X\) satisfies

\[
\psi(d(Tx, Ty)) \leq \psi(m(x, y)) - \phi(d(x, y))
\]

(2.13)

for all \(x, y \in X\).

Then \(T\) has a fixed point in \(X\).

**Proof.** Let \(\alpha : X \times X \rightarrow [0, \infty)\) be a function defined by \(\alpha(x, y) = 1\) for all \(x, y \in X\).

Then all conditions of Theorem 2.1 are satisfied. Hence, we have the desired result. \(\square\)

**Corollary 2.6.** Let \((X, d)\) be a complete cone metric space with regular cone \(P\) such that \(d(x, y) \in \text{int}(P)\) for \(x, y \in X\) with \(x \neq y\). Suppose that a mapping \(T : X \rightarrow X\) satisfies

\[
\psi(d(Tx, Ty)) \leq \psi(n(x, y)) - \phi(d(x, y))
\]

for all \(x, y \in X\).

Then \(T\) has a fixed point in \(X\).

**Corollary 2.7.** Let \((X, d)\) be a complete cone metric space with regular cone \(P\) such that \(d(x, y) \in \text{int}(P)\) for \(x, y \in X\) with \(x \neq y\). Suppose that a mapping \(T : X \rightarrow X\) satisfies

\[
\psi(d(Tx, Ty)) \leq \psi(n(x, y)) - \phi(n(x, y))
\]

for all \(x, y \in X\), where \(\phi \in \Phi\) is nondecreasing.

Then \(T\) has a fixed point in \(X\).

**Remark 2.2.** (1) Corollary 2.5 is a generalization of Theorem 2.1 in [14]. In fact, if we have \(\psi(t) = t\) for all \(t \in P\), and \(p = 1, q = r = s = 0\) then Corollary 2.5 reduces to Theorem 2.1 in [14].

(2) If \(E = \mathbb{R}, P = [0, \infty), p = 1\) and \(q = r = s = 0\), then Corollary 2.6 reduces to Theorem 2.1 in [17].

(3) Let \(\psi(t) = t\) for all \(t \in P\) and \(\phi(t) = \frac{1}{2}t\) for all \(t \in \text{int}(P) \cup \{0\}\), and let \(\varphi(t) = t - \phi(t)\) for all \(t \in \text{int}(P) \cup \{0\}\).

Then, Corollary 2.7 becomes to Corollary 2.3 in [2].

**Corollary 2.8.** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a cone metric \(d\) on \(X\) such that \((X, d)\) is a complete cone metric space with regular cone \(P\) such that \(d(x, y) \in \text{int}(P)\) for \(x, y \in X\) with \(x \neq y\). Suppose that a mapping \(T : X \rightarrow X\) satisfies the following conditions:
Fixed point theorem

(1) \( \psi(d(Tx, Ty)) \leq \psi(m(x, y)) - \phi(d(x, y)) \)
for all \( x, y \in X \) with \( x \preceq y \);

(2) there exists \( x_1 \in X \) such that \( x_1 \preceq Tx_1 \);

(3) \( T \) is nondecreasing;

(4) either \( T \) is continuous or if \( \{x_n\} \subset X \) is a sequence with \( x_n \preceq x_{n+1} \) for all \( n \in \mathbb{N} \) and \( x \) is a cluster point of \( \{x_n\} \), then there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( x_{n(k)} \preceq x \) for all \( k \in \mathbb{N} \).

Then \( T \) has a fixed point in \( X \).

Proof. Define a function \( \alpha : X \times X \to [0, \infty) \) by

\[
\alpha(x, y) = \begin{cases} 
1 & \text{if } x \preceq y, \\
0 & \text{otherwise.}
\end{cases}
\]

Then, from (1) we have \( \alpha(x, y) \psi(d(Ty, Tx)) \leq \psi(m(x, y)) - \phi(d(x, y)) \) for all \( x, y \in X \), and so \( T \) is a generalized weak \( \alpha \)-contraction.

Obviously, condition (1) of Theorem 2.1 is satisfied. Since \( T \) is nondecreasing, \( \alpha(x, y) = 1 \) implies \( \alpha(Tx, Ty) = 1 \) for all \( x, y \in X \). Thus, the condition (2) of Theorem 2.1 is satisfied.

Condition (2) implies that there exists \( x_1 \in X \) such that \( \alpha(x_1, Tx_1) = 1 \), and so condition (3) of Theorem 2.1 is satisfied.

From definition of the function \( \alpha \), condition (4) implies condition (4) of Theorem 2.1.

Thus, all conditions of Theorem 2.1 are satisfied. From Theorem 2.1 \( T \) has a fixed point in \( X \).

Corollary 2.9. Let \( (X, \preceq) \) be a partially ordered set and suppose that there exists a cone metric \( d \) on \( X \) such that \((X, d)\) is a complete cone metric space with regular cone \( P \) such that \( d(x, y) \in \text{int}(P) \) for \( x, y \in X \) with \( x \neq y \). Suppose that a mapping \( T : X \to X \) satisfies conditions (1), (2) and (3) in Corollary 2.8.

Assume that either \( T \) is continuous or if \( \{x_n\} \subset X \) is a sequence with \( x_n \preceq x_{n+1} \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} x_n = x \), then \( x_n \preceq x \) for all \( n \in \mathbb{N} \).

Then \( T \) has a fixed point in \( X \).

Corollary 2.10. Let \( (X, \preceq) \) be a partially ordered set and suppose that there exists a cone metric \( d \) on \( X \) such that \((X, d)\) is a complete cone metric space with regular cone \( P \) such that \( d(x, y) \in \text{int}(P) \) for \( x, y \in X \) with \( x \neq y \).

Let \( T : X \to X \) be a mapping such that

\[
\psi(Tx, Ty)) \leq \psi(n(x, y)) - \phi(d(x, y))
\]
for all \( x, y \) with \( x \preceq y \).

Suppose that \( T : X \to X \) satisfies conditions (2), (3) and (4) in Corollary 2.8.

Then \( T \) has a fixed point in \( X \).

**Corollary 2.11.** [16] Let \((X,\preceq)\) be a partially ordered set and suppose that there exists a cone metric \( d \) on \( X \) such that \((X,d)\) is a complete cone metric space with regular cone \( P \) such that \( d(x,y) \in \text{int}(P) \) for \( x, y \in X \) with \( x \neq y \).

Let \( T : X \to X \) be a mapping such that

\[
\psi(Tx,Ty)) \leq \psi(n(x,y)) - \phi(d(x,y))
\]

for all \( x, y \) with \( x \preceq y \).

Suppose that \( T : X \to X \) satisfies conditions (2) and (3) in Corollary 2.8. Assume that either \( T \) is continuous or if \( \{x_n\} \) is a sequence with \( x_n \preceq x_{n+1} \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} x_n = x \), then \( x_n \preceq x \) for all \( n \in \mathbb{N} \).

Then \( T \) has a fixed point in \( X \).

**Corollary 2.12.** Let \((X,\preceq)\) be a partially ordered set and suppose that there exists a cone metric \( d \) on \( X \) such that \((X,d)\) is a complete cone metric space with regular cone \( P \) such that \( d(x,y) \in \text{int}(P) \) for \( x, y \in X \) with \( x \neq y \).

Let \( T : X \to X \) be a mapping such that

\[
\psi(Tx,Ty)) \leq \psi(n(x,y)) - \phi(d(x,y))
\]

for all \( x, y \) with \( x \preceq y \).

Suppose that \( T : X \to X \) satisfies conditions (2) and (3) in Corollary 2.8. Assume that either \( T \) is continuous or if \( \{x_n\} \) is a sequence with \( x_n \preceq x_{n+1} \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} x_n = x \), then \( x_n \preceq x \) for all \( n \in \mathbb{N} \).

Then \( T \) has a fixed point in \( X \).

**Remark 2.3.** Let \( \psi(t) = t \) for all \( t \in P \) and \( \phi(t) = \frac{1}{2} t \) for all \( t \in \text{int}(P) \cup \{0\} \), and \( \psi(t) = t - \phi(t) \) for all \( t \in \text{int}(P) \cup \{0\} \).

Then Corollary 2.12 reduces to Theorem 12 and Theorem 13 in [4].

### 3 Application to integral equations

Let \( E = \mathbb{R}^2, \) \( P = \{(u,v) \in \mathbb{R}^2 : u,v \geq 0\}, \) \( X = C(\mathbb{R}_+,\mathbb{R}), \) and let \( d(x,y) = (\|x-y\|_{\infty},\|x-y\|_{\infty}) \) for all \( x, y \in X \).

It is easy to see that \((X,d)\) is a complete cone metric space and \( P \) is regular.

Consider the following Urysohn integral equation:

\[
x(t) = g(t) + \int_0^\infty u(t,s,x(s))ds \quad (3.1)
\]

where \( t \geq 0, x, g \in X \) and \( u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) is a continuous function.

We consider the following conditions.
(A) for each $x \in X$
\[
\int_0^\infty u(t, s, x(s))ds \in X;
\]

(B) there exists a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that the following are satisfied:

(i) for all $x, y \in X$ and for all $t \geq 0$
\[
\xi(x(t), y(t)) \geq 0 \text{ and } \xi(y(t), z(t)) \geq 0 \implies \xi(x(t), z(t)) \geq 0;
\]

(ii) there exists a continuous function $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that
\[
p := \sup_{t \geq 0} \int_0^\infty k(t, s)ds < 1
\]
and
\[
| u(t, s, x(s)) - u(t, s, y(s)) | \leq k(t, s) \ln(| x(s) - y(s) | + 1)
\]
for all $x, y \in X$ with $\xi(x(t), y(t)) \geq 0$ for all $t \geq 0$;

(iii) there exists $x_1 \in X$ such that for all $t \geq 0$
\[
\xi(x_1(t), \int_0^\infty u(t, s, x_1(s))ds) \geq 0;
\]

(iv) for all $x, y \in X$ and for all $t \geq 0$
\[
\xi(x(t), y(t)) \geq 0 \text{ implies } \xi(\int_0^\infty u(t, s, x(s))ds, \int_0^\infty u(t, s, y(s))ds) \geq 0;
\]

(v) If $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} x_n = x \in X$ and $\xi(x_n, x_{n+1}) \geq 0$ for all $n \in \mathbb{N}$, then $\xi(x_n, x) \geq 0$ for all $n \in \mathbb{N}$.

**Theorem 3.1.** If conditions (A) and (B) are satisfied, then the Urysohn integral equation (3.1) has at least one solution $x_* \in X$.

**Proof.** Define a mapping $T : X \to X$ by
\[
Tx(t) = \int_0^\infty u(t, s, x(s))ds \text{ for all } t \geq 0.
\]
Let $x, y \in X$ be such that $\xi(x(t), y(t)) \geq 0$ for all $t \geq 0$. 

From (ii) we have
\[
\left| Tx(t) - Ty(t) \right| \\
= \left| \int_0^\infty [u(t, s, x(s)) - u(t, s, y(s))] ds \right| \\
\leq \int_0^\infty | u(t, s, x(s)) - u(t, s, y(s)) | ds \\
\leq \int_0^\infty k(t, s) \ln(\| x(s) - y(s) \| + 1) ds \\
\leq \sup_{t \geq 0} \int_0^\infty k(t, s) ds \ln(\| x(s) - y(s) \|_\infty + 1) \\
= p \ln(\| x(s) - y(s) \|_\infty + 1) \\
\leq p \| x - y \|_\infty - (p \| x - y \|_\infty - p \ln(\| x(s) - y(s) \|_\infty + 1)) \\
= p \| x - y \|_\infty - \phi_1(\| x - y \|_\infty) \\
\leq m_1(x, y) - \phi_1(\| x - y \|_\infty),
\]
where \( m_1(x, y) = p \| x - y \|_\infty + q \| x - Tx \|_\infty + r \| y - Ty \|_\infty + s \| x - Ty \|_\infty + \| y - Tx \|_\infty \), and \( p := \sup_{t \geq 0} \int_0^\infty k(t, s) ds < 1 \), and \( p + q + r + 2s \leq 1 \) and \( \phi_1(t) = pt - p \ln(t + 1) \).

Then, we have \( \| Tx - Ty \|_\infty \leq m_1(x, y) - \phi_1(\| x - y \|_\infty) \) for all \( x, y \in X \) with \( \xi(x(t), y(t)) \geq 0 \) for all \( t \geq 0 \).

Let \( \phi_1(v) = \phi_2(v) \) for all \( v \geq 0 \), and let \( \phi(v, w) = (\phi_1(v), \phi_2(w)) \) for all \( v, w \geq 0 \).

Then, \( \phi \in \Phi \).

Let \( m_1(x, y) = m_2(x, y) \) for all \( x, y \in X \), and let \( m(x, y) = (m_1(x, y), m_2(x, y)) \).

Then, we have \( d(Tx, Ty) \leq m(x, y) - \phi(d(x, y)) \) for all \( x, y \in X \) such that \( \xi(x(t), y(t)) \geq 0 \) for all \( t \geq 0 \).

Define a function \( \alpha : X \times X \to [0, \infty) \) by
\[
\alpha(x, y) = \begin{cases} 
1, & \text{if } \xi(x(t), y(t)) \geq 0, t \geq 0, \\
0, & \text{otherwise} .
\end{cases}
\]

Then, for all \( x, y \in X \), we have
\[
\alpha(x, y)d(Tx, Ty) \leq m(x, y) - \phi(d(x, y))
\]
and so \( T \) is a generalized weakly \( \alpha \)-contractive mapping defined on \( (X, d) \).

Obviously, \( \alpha(x, y) \geq 1 \) and \( \alpha(y, z) \geq 1 \) implies \( \alpha(x, z) \geq 1 \) for all \( x, y, z \in X \).

If \( \alpha(x, y) \geq 1 \) for all \( x, y \in X \), then \( \xi(x(t), y(t)) \geq 0 \). From (iv) we have \( \xi(Tx(t), Ty(t)) \geq 0 \), and so \( \alpha(Tx, Ty) \geq 1 \).

From (iii) there exists \( x_1 \in X \) such that \( \alpha(x_1, Tx_1) \geq 1 \).
From (v) we have that if \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} x_n = x \in X \) and \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \), then \( \alpha(x_n, x) \geq 1 \) for all \( n \in \mathbb{N} \).

By applying Corollary 2.3 with \( \psi(t) = t \) for all \( t \in P \), \( T \) has a fixed point in \( X \), and hence (3.1) has a solution.

\[ \square \]

**Acknowledgement**

This research (Ji-Young Lee) was supported by Hanseo University.

**References**


Received: March 11, 2014