Synchronization of Unsmooth Chua Chaotic Circuits
Based on Numerical Optimization

Hong-Bin Bai

School of Science, Sichuan University of Science and Engineering,
Zigong, Sichuan, 643000, P. R. China

Shu-Lin Wu

School of Science, Sichuan University of Science and Engineering,
Zigong, Sichuan, 643000, P. R. China

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Abstract

In the field of secure communication, chaos and its synchronization are the key problems. In this paper, we focus on designing an efficient synchronization controller to synchronize two Chua’s chaotic circuits with nonsmooth nonlinear term $x|x|$. A master-slave synchronization controller based on master-slave scheme is designed for the underlying Chua’s circuits. Based on this scheme, some sufficient criteria for global synchronization are proved. The controller contains a free parameter and we use numerical optimization to optimize it. The effectiveness of the obtained criteria is illustrated by numerical simulations.

Mathematics Subject Classification: Primary 47B38; Secondary 47B33, 47B37

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1 Introduction

In 1983, Leon O. Chua first proposed a simple circuit which was later named after his name Chua and can be regarded as one of the most important
circuits in many areas [1]: Since the pioneering work by Pecora and Carroll, who originally proposed the master Cslave (also called drive-response) concept for achieving the synchronization of chaotic systems [2], [3], researchers have proposed a variety of schemes for ensuring the control and synchronization of such systems due to its potential applications for image processing, secure communication, information science, and harmonic oscillation generation [4]-[6]. Among these studies, the Chua’s chaotic system has attracted lots of attention due to its representation and importance in many areas [7]-[10].

It is known long that many nonlinear functions (or say maps) can generate chaos, such as the smooth quadratic functions (e.g. Lorenz system with the Henon map), the piecewise linear functions (e.g. Chua’s circuit with the Lozi map) and the cubic function (e.g., the Duffing oscillator). It is therefore very natural to ask: if a piecewise quadratic function such as $x|x|$ can play the same role as a chaos generator. In [11], the authors presented a positive answer for this issue with a rigorous mathematical verification of the generated chaos (by means of proving the existence of the Smale horseshoes), thereby providing a more complete scenario of chaos generation via lower-order polynomial nonlinearities. The circuit implementation and bifurcation analysis of the modified Chua’s circuit with $x|x|$ function was reported in [12]. The synchronization problem of two coupled Chua’s chaotic systems with nonsmooth nonlinear term $x-x$ has also been studied by some researchers in recent years [13] and [14]. In these papers, the authors utilize the nonlinear feedback control strategy to synchronize the drive and the response systems.

The goal of this paper is to present a more efficient synchronization controller to synchronize the two Chua’s chaotic circuits with nonsmooth nonlinear term $x|x|$. The controller consists of a master-slave synchronizing scheme and a simple linear feedback control. The usage of proposed linear feedback control strategy ensues that the presented synchronization controller is simple.
and easy to use, compared to the nonlinear feedback control. The synchronization criterion for global synchronization is presented by using Lyapunov exponential stability principle. The synchronization controller proposed here contains a free parameter which can be used to maximize the synchronization rate. Through numerical optimization we show that by properly choosing the parameter, the synchronization rate can be significantly improved. Finally, a couple of numerical results are given to validate the effectiveness of the proposed synchronization controller.

2 THE MODEL CIRCUIT AND THE SYNCHRONIZATION CONTROLLER

Our model circuit studied in this paper is

\[
\begin{align*}
    x'(t) &= \alpha(y(t) + ax(t) - bx(t)|x(t)|), \\
    y'(t) &= x(t) - y(t) + z(t), \\
    z'(t) &= -\beta y(t),
\end{align*}
\]

where \(\alpha, \beta, a, b\) are positive constants. We regard (1) as the drive system and the response system is defined by

\[
\begin{align*}
    \tilde{x}'(t) &= \alpha(\tilde{y}(t) + a\tilde{x}(t) - b\tilde{x}(t)|\tilde{x}(t)|), \\
    \tilde{y}'(t) &= \tilde{x}(t) - \tilde{y}(t) + \tilde{z}(t), \\
    \tilde{z}'(t) &= -\beta \tilde{y}(t).
\end{align*}
\]

To synchronize the drive and the response systems, we must impose some control strategy to the response system (2). Here, we consider the following feedback control strategy

\[
\begin{align*}
    \tilde{x}'(t) &= \alpha(\tilde{y}(t) + a\tilde{x}(t) - b\tilde{x}(t)|\tilde{x}(t)|) + C_1(t), \\
    \tilde{y}'(t) &= \tilde{x}(t) - \tilde{y}(t) + \tilde{z}(t) + C_2(t), \\
    \tilde{z}'(t) &= -\beta \tilde{y}(t) + C_3(t),
\end{align*}
\]

where the control variables \(C_j(t)\) are defined by

\[
\begin{align*}
    C_1(t) &= \gamma e_x(t), \\
    C_2(t) &= \gamma e_x(t) + e_z(t), \\
    C_3(t) &= -\beta e_y(t) + \gamma e_z(t),
\end{align*}
\]

where \(e_x(t) = x(t) - \tilde{x}(t)\)(and similarly \(e_y(t) = y(t) - \tilde{y}(t)\) and \(e_z(t) = z(t) - \tilde{z}(t)\)). The parameter \(\gamma\) involved in (4) is free and can be optimized the synchronization rate. In the sequel, we focus on optimizing \(\gamma\) by maximizing the synchronization rate.
3 THEORETICAL ANALYSIS

In this section, we analyze the synchronization controller presented in (4). We focus on proving the global exponential synchronization of the drive system (1) and the response system (3). The following lemma is necessary to realize this goal.

**Lemma 1([14]):** Define \(e_x = x - \tilde{x}\) and

\[
F(e_x) = x|x| - \tilde{x}|	ilde{x}|
\]

(5)

We have \(e_x f(e_x) \geq 0\) and ‘=’ holds if and only if \(e_x = 0\).

**Proof:** The function \(F\) is the standard feedback function of the Lurie control system. The proof of this lemma can be found in [14]. Based on the aforementioned lemma, we now present the main result of this paper.

**Theorem 1:** Suppose the parameter \(\gamma\) satisfies

\[
\min_{j=1,2,3} \lambda_j(\bar{A}) > 0.
\]

(6)

where \(\bar{A} = \begin{pmatrix}
2(\gamma - \alpha a) & \gamma - 1 - \alpha & 0 \\
\gamma - 1 - \alpha & 2 & 0 \\
0 & 0 & 2\gamma
\end{pmatrix}\) and \(\lambda_j(\bar{A})\) denotes the \(j\)-th eigenvalue of the matrix \(\bar{A}\). Then for any initial values \((x(0), y(0), z(0))^T\) and \((\tilde{x}(0), \tilde{y}(0), \tilde{z}(0))^T\), the drive and the response systems (1) and (3) can be exponentially synchronized. In particular, the error vector \(E(t) = (e_x(t), e_y(t), e_z(t))^T\) satisfies

\[
\|E(t)\|_2 \leq \|E(0)\|_2 e^{-\theta t},
\]

(7)

where \(k\) is the so-called synchronization rate, which is defined by

\[
\theta = \frac{\min_{j=1,2,3} \lambda_j(\bar{A})}{2}.
\]

(8)

**Proof:** We first note that the error functions between the drive and the response systems satisfy

\[
\begin{align*}
e'_x(t) &= \alpha[e_x(t) + ae_x(t) - bF(e_x(t))] - \gamma e_x(t), \\
e'_y(t) &= e_x(t) - e_y(t) - \gamma e_x(t), \\
e'_z(t) &= -\gamma e_z(t).
\end{align*}
\]

(9)
Let \( V(t) = e^{\kappa t} \|E(t)\|_2^2 \) and it holds

\[
\frac{dV(t)}{dt} = \kappa e^{\kappa t} \|E(t)\|_2^2 + 2e^{\kappa t} E^T(t) E'(t)
\]

\[
= e^{\kappa t}(\kappa \|E(t)\|_2^2 + 2e_x(t) |ae_y(t) + (\alpha a - \gamma) e_x(t) - 2\alpha e_x(t) F(e_x(t)) - 2(1 - \gamma) e_x(t) e_y(t) - 2\gamma e^2_x(t))
\]

\[
- 2\alpha e_x(t) F(e_x(t)) - 2(1 - \gamma) e_x(t) e_y(t) - 2\gamma e^2_x(t)
\]

\[
\leq e^{\kappa t}(\kappa \|E(t)\|_2^2 + 2(\alpha a - \gamma) e^2_x(t) - 2e^2_y(t))
\]

\[
- 2\gamma e^2_x(t) + 2(\alpha + 1 - \gamma) e_x(t) e_y(t),
\]  

(10)

where we have used Lemma 1 and the fact that \( \alpha, b > 0 \). The inequality (10) can be rewritten concisely as

\[
E'(t) = e^{\kappa t} E^T(t) A E(t),
\]

(11)

where \( A = \begin{pmatrix} \kappa - 2(\gamma - \alpha a) & \alpha + 1 - \gamma & 0 \\ \alpha + 1 - \gamma & \kappa - 2 & 0 \\ 0 & 0 & \kappa - 2\gamma \end{pmatrix} = \kappa I - \bar{A}. \)

Clearly, \( \bar{A} \) is a real symmetrical matrix and therefore all its eigenvalues are real numbers. Under the condition that \( \bar{A} \) is positive definite, i.e., all its eigenvalues are positive, we can select some \( \kappa \) such that

\[
0 < \kappa \leq \min_{j=1,2,3} \lambda_j(\bar{A}).
\]

(12)

Under this condition, it is easy to know that the matrix \( A \) is semi-negative definite. Hence, we get \( V'(t) \leq 0 \) and this implies that \( V(t) \leq V(0) \). We therefore arrive at

\[
\|E(t)\|_2 \leq e^{-\theta t} \|E(0)\|,
\]

and the largest \( \theta \) is defined by (8).

Since \( \gamma \) is a free parameter, the maximal synchronization rate is defined by the following max-min problem

\[
\theta_{\text{max}} = \max_{\gamma} \min_{j} \lambda_j(\bar{A}).
\]

(13)

Completely solving the max-min problem (13) is difficult and obviously beyonds the scope of this paper. In the sequel, we give some results based on numerical optimization. In Figure 2, we plot \( \max_j \lambda_j(\bar{A}) \) as a function of \( \gamma \) for a given circuit parameters \((\alpha, \beta, a, b) = (5.982, 6065, 0.0516432, 1.5 \times 10^{-4})\), where we can see that there exists an optimal choice of the free parameter \( \gamma_{\text{opt}} = 6.975 \) and under this choice the maximal synchronization rate is \( \theta_{\text{max}} = 1. \)
Fig 3. Without synchronization control, the phase figure of the drive system (top) and the response system (bottom).

4 NUMERICAL SIMULATIONS

In this section, we provide numerical results to validate the efficiency of the synchronization controller (4). Let \((x(0), y(0), x(0))^T = (0.3, -0.1, 0.3)^T\), \((\tilde{x}(0), \tilde{y}(0), \tilde{x}(0))^T = (0.7, 0.3, 0.2)^T\) and \(a, \beta, a, b = (5.982, 6.65, 0.0516432, 1.5 \times 10^{-4})\) and then we plot in Figure 3 the phase figure of the drive and response system without control. We see clearly in this figure that drive system and the response system (without control) behave differently.

We now impose the synchronization controller (4) to the response system and with three different choices of \(\gamma\) (the optimal choice of \(\gamma\) is 6.975). Then, we plot in Figures 4-6 the dynamical behavior of the derive system and the response system with three choices of the parameter \(\gamma\). By comparing the results plotted in these three figures, one can see clearly that synchronization effect with \(\gamma = 6\) and \(\gamma = 13\) has only slight difference and both are poor compared to the optimal choice \(\gamma = 6.975\) is the optimal choice. We now measure the synchronization rate of the drive and the response systems under the three choices of \(\gamma\) used in the above experiments. To this end, we plot in figure 7 the error \(\|E(t)\|_2\) between the drive and the response systems as
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Fig4. Time dynamical behavior of the solution component $x_1(t)$ (derive system) and $y_1(t)$ (response system).

Fig5. Time dynamical behavior of the solution component $x_2(t)$ (derive system) and $y_2(t)$ (response system).

Fig6. Time dynamical behavior of the solution component $x_3(t)$ (derive system) and $y_3(t)$ (response system).

Fig7. Synchronization rate under three different choices of the free parameter $\gamma$. 
a function of $t$. It is clear that the choice $\gamma = 6.975$ is much more advisable, compared to the other two choices of $\gamma : \gamma = 3$ and $\gamma = 11$. In particular, to achieve synchronization effect $10^{-12}$, the optimal synchronization controller with $\gamma = 6.975$ only needs 7 seconds, while the other ones need about 14 seconds. Again, this observation coincides with the results given in Figure 2.

5 CONCLUSION

We have designed a new synchronization controller which only linear feedback control for the Chua’s chaotic circuits with unsmooth nonlinear term $x \mid x \mid$. Global exponential synchronization is investigated from theoretical analysis and numerical optimization aspects. It is shown that, by properly chosen the free parameter involved in the controller, the synchronization controller results in much faster synchronization rate. Numerical simulations are provided to support out conclusion and it is shown that the numerical optimization really gives more efficient parameter $\gamma$ involved in the synchronization controller. Further work including solve the max-min problem at theoretical level and generalize the current work to more complex and large scale system of IDE’s.

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