On the Shape Gradient and Shape Hessian of a Shape Functional Subject to Dirichlet and Robin Conditions

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Abstract

This paper focuses on minimizing a shape functional through the solution of a Pure Dirichlet boundary value problem, and a Dirichlet-Robin boundary value problem. This shape optimization problem is a variant of the Kohn-Vogelius shape optimization formulation of a Bernoulli free boundary problem. The first- and second-order shape derivatives of the cost functional under consideration are explicitly derived. Interestingly, the present findings coincide with the existing results regarding solutions to the Bernoulli problem.

Keywords: shape gradient, shape Hessian, Kohn-Vogelius objective functional, Dirichlet boundary value problem, Robin boundary value problem

1 Introduction

The present paper derives the shape gradient and shape Hessian of the functional $J$ in the minimization problem

$$\min_{\Omega} J(\Omega) \equiv \min_{\Omega} \int_{\Omega} |\nabla(u_D - u_N)|^2 \, dx$$

(1)
where the state functions \( u_D \) and \( u_N \) satisfy the following Dirichlet and Robin boundary value problems, respectively:

\[
\begin{align*}
-\Delta u_D &= 0 \quad \text{in } \Omega, \\
u_D &= 1 \quad \text{on } \Gamma, \\
u_D &= 0 \quad \text{on } \Sigma.
\end{align*}
\]

(2)

\[
\begin{align*}
-\Delta u_N &= 0 \quad \text{in } \Omega, \\
u_N &= 1 \quad \text{on } \Gamma, \\
\alpha u_N + \frac{\partial u_N}{\partial n} &= \lambda \quad \text{on } \Sigma,
\end{align*}
\]

(3)

where \( \alpha \geq 0 \) is fixed, and \( \lambda < 0 \).

The shape optimization formulation (1) subject to (2) and (3) is derived from the two-dimensional exterior Bernoulli free boundary problem, a problem wherein we are given a constant \( \lambda < 0 \) and a bounded and connected domain, say \( A \subset \mathbb{R}^2 \) with a fixed boundary \( \Gamma := \partial A \), and our task is to find a bounded connected domain \( B \subset \mathbb{R}^2 \) with a free boundary \( \Sigma \) and containing the closure of \( A \), as well as a state function \( u : \Omega \to \mathbb{R} \), where \( \Omega = B \setminus \bar{A} \), that satisfies the following boundary value problem

\[
\begin{align*}
-\Delta u &= 0 \quad \text{in } \Omega, \\
u &= 1 \quad \text{on } \Gamma, \\
u = 0, \frac{\partial u}{\partial n} &= \lambda \quad \text{on } \Sigma,
\end{align*}
\]

(4)

where \( n \) is the outward unit normal vector to \( \Sigma \).

The present study is motivated by the work of Tiihonen [9] where he computed the shape gradient and shape Hessian of a different functional formulation of (4). In [9], Tiihonen considered the following shape optimization formulation:

\[
\min_{\Sigma} J(\Sigma) = \min_{\Sigma} \int_{\Sigma} u_N^2 \, ds
\]

(5)

where \( u_N \) satisfies the conditions (3).

2 Preliminaries

The paper requires the following results and tools from shape calculus. These are found in [1, 3]:

**Theorem 2.1.** Let \( \Omega \) and \( U \) be nonempty bounded open connected subsets of \( \mathbb{R}^2 \) with Lipschitz continuous boundaries, such that \( \bar{\Omega} \subseteq U \), and \( \partial \Omega \) is the union of two disjoint boundaries \( \Gamma \) and \( \Sigma \). Let \( T_t \) be defined as

\[
T_t : \bar{U} \to \mathbb{R}^2, \quad T_t(x) = x + tV(x), \quad x \in \bar{U},
\]

(6)
where \( \mathbf{V} \) belongs to \( \Theta \), defined as
\[
\Theta = \{ \mathbf{V} \in C^{1,1}(\bar{U}, \mathbb{R}^2) : \mathbf{V}|_{\Gamma \cup \partial U} = 0 \}.
\] (7)

Then for sufficiently small \( t \),
\[
(1.) T_t : \bar{U} \to \bar{U} \text{ is a homeomorphism}, \quad (4.) \Gamma_t = T_t(\Gamma) = \Gamma,
\]
\[
(2.) T_t : U \to U \text{ is a } C^{1,1} \text{ diffeomorphism}, \quad (5.) \Sigma_t = T_t(\Sigma), \text{ and}
\]
\[
(3.) T_t : \Omega \to \Omega_t \text{ is a } C^{1,1} \text{ diffeomorphism}, \quad (6.) \partial \Omega_t = \Gamma \cup \Sigma_t.
\]

For the following functions
\[
\begin{align*}
I_t(x) &= \det DT_t(x), \quad x \in \bar{U}, \\
M_t(x) &= (DT_t(x))^{-T}, \quad x \in \bar{U}, \\
A_t(x) &= I_t M_t^T M_t(x), \quad x \in \bar{U}, \\
w_t(x) &= I_t(x)| DT_t(x)|^{-T} n(x), \quad x \in \Sigma
\end{align*}
\] (8)
we have the following lemma:

**Lemma 2.2.** [7, 8] Consider the transformation \( T_t \), where the fixed vector field \( \mathbf{V} \) belongs to \( \Theta \), defined in (7). Then there exists \( t_V > 0 \) such that \( T_t \) and the functions in (8) restricted to the interval \( I_V = (-t_V, t_V) \) have the following regularity and properties:

\[
\begin{align*}
(1.) & \ t \mapsto T_t \in C^1(I_V, C^{1,1}(\bar{U}, \mathbb{R}^2)). & \quad (8.) & \frac{d}{dt} T_t^{-1}|_{t=0} = -\mathbf{V}.
\end{align*}
\]
\[
\begin{align*}
(2.) & \ t \mapsto I_t \in C^1(I_V, C^{0,1}(\bar{U})). & \quad (9.) & \frac{d}{dt} DT_t|_{t=0} = D\mathbf{V}.
\end{align*}
\]
\[
\begin{align*}
(3.) & \ t \mapsto T_t^{-1} \in C(I_V, C^{1}(\bar{U}, \mathbb{R}^2)). & \quad (10.) & \frac{d}{dt} (DT_t)^{-1}|_{t=0} = -D\mathbf{V}.
\end{align*}
\]
\[
\begin{align*}
(4.) & \ t \mapsto w_t \in C^1(I_V, C(\Sigma)). & \quad (11.) & \frac{d}{dt} I_t|_{t=0} = \text{div } \mathbf{V}.
\end{align*}
\]
\[
\begin{align*}
(5.) & \ t \mapsto A_t \in C(I_V, C(\bar{U}, \mathbb{R}^{2\times2})). & \quad (12.) & \frac{d}{dt} A_t|_{t=0} = A,
\end{align*}
\]
\[
\begin{align*}
(6.) & \text{There is } \beta > 0 \text{ such that } A_t(x) \geq \beta I \text{ for } x \in U. & \quad (13.) & \lim_{t \to 0} w_t = 1.
\end{align*}
\]
\[
\begin{align*}
(7.) & \frac{d}{dt} T_t|_{t=0} = \mathbf{V}. & \quad (14.) & \frac{d}{dt} w_t|_{t=0} = \text{div}_\Sigma \mathbf{V}
\end{align*}
\]
\[
\]
\[
\text{where } \text{div}_\Sigma \mathbf{V} = \text{div } \mathbf{V}|_\Sigma - (D\mathbf{V}n) \cdot n.
\]

**Material and shape derivatives of states**

**Definition 2.3.** Let \( u \) be defined in \( [0, t_V] \times U \). The material derivative \( \dot{u} \in H^k(\Omega) \) of \( u \) is defined as
\[
\dot{u}(x) := \dot{u}(0, x) := \lim_{t \to 0^+} \frac{u(t, T_t(x)) - u(0, x)}{t} = \frac{d}{dt} u(t, x + t\mathbf{V}(x))|_{t=0}
\]
if the limit exists in \( H^k(\Omega) \).
It can also be written as
\[ \dot{u}(x) = \lim_{t \to 0^+} \frac{u_t \circ T_t(x) - u(x)}{t} = \frac{d}{dt}(u_t \circ T_t(x)) \bigg|_{t=0}. \] (9)

**Definition 2.4.** Let \( u \) be defined in \([0, t_V] \times U \). The shape derivative \( u' \in H^k(\Omega) \) of \( u \) is defined as:

\[ u'(x) := u'(0, x) := \lim_{t \to 0^+} \frac{u(t, x) - u(0, x)}{t}. \] (10)

if the limit exists in \( H^k(\Omega) \).

It can also be written as
\[ u'(x) = \dot{u}(x) - (\nabla u \cdot \mathbf{V})(x). \] (11)

**Domain and boundary transformations**

**Lemma 2.5.** [10]

1. Let \( \varphi_t \in L^1(\Omega_t) \). Then \( \varphi_t \circ T_t \in L^1(\Omega) \) and \( \int_{\Omega_t} \varphi_t \, dx_t = \int_{\Omega} \varphi_t \circ T_t I_t \, dx \).

2. Let \( \varphi_t \in L^1(\partial \Omega_t) \). Then \( \varphi_t \circ T_t \in L^1(\partial \Omega) \) and \( \int_{\partial \Omega_t} \varphi_t \, ds_t = \int_{\partial \Omega} \varphi_t \circ T_t w_t \, ds \).

where \( I_t \) and \( w_t \) are defined in (8).

**Some tangential Calculus**

Here are some properties of tangential differential operators which are used in this work (cf. [4, 10]). Let \( \Gamma \) be a boundary of a bounded domain \( \Omega \subset \mathbb{R}^n \).

**Definition 2.6.** The tangential gradient of \( f \in C^1(\Gamma) \) is given by
\[ \nabla_{\Gamma} f := \nabla F|_{\Gamma} - \frac{\partial F}{\partial \mathbf{n}} \mathbf{n} \in C(\Gamma, \mathbb{R}^n), \] (12)

where \( F \) is any \( C^1 \) the extension of \( f \) into a neighborhood of \( \Gamma \).

**Definition 2.7.** The tangential Jacobian matrix of a vector function \( \mathbf{v} \in C^1(\Gamma, \mathbb{R}^n) \) is given by
\[ D_{\Gamma} \mathbf{v} = D \mathbf{V}|_{\Gamma} - (D \mathbf{V} \mathbf{n})^T \mathbf{n} \in C(\Gamma, \mathbb{R}^{n \times n}), \] (13)

where \( \mathbf{V} \) is any \( C^1 \) the extension of \( \mathbf{v} \) into a neighborhood of \( \Gamma \).

**Definition 2.8.** For a vector function \( \mathbf{v} \in C^1(\Gamma, \mathbb{R}^n) \), its tangential divergence on \( \Gamma \) is given by
\[ \text{div}_{\Gamma} \mathbf{v} = \text{div} \mathbf{V}|_{\Gamma} - D \mathbf{V} \mathbf{n} \cdot \mathbf{n} \in C(\Gamma), \] (14)

where \( \mathbf{V} \) is any \( C^1 \) the extension of \( \mathbf{v} \) into a neighborhood of \( \Gamma \).
Shape Differentiation of Integrals
Let \( u \in L^1(\Omega) \). Suppose there exist \( \dot{u} \in L^1(\Omega) \) and \( u' \in L^1(\Omega) \). Then for sufficiently smooth \( \Omega \) and \( V \),

\[
\frac{d}{dt} \int_{\Omega_t} u(t, x) \, dx \bigg|_{t=0} = \int_{\Omega} u'(0, x) \, dx + \int_{\partial\Omega} u(0, s) V \cdot n \, ds \tag{15}
\]

Similarly, if \( u \in L^1(\Gamma) \) and there exist \( \dot{u} \in L^1(\Gamma) \) and \( u' \in L^1(\Gamma) \), then

\[
\frac{d}{dt} \int_{\Gamma_t} u(t, s) \, ds \bigg|_{t=0} = \int_{\Gamma} u'(0, s) \, ds + \int_{\Gamma} (\frac{\partial u}{\partial n} + u(0, s) \kappa) V \cdot n \, ds \tag{16}
\]

where \( \kappa \) is the mean curvature of the boundary \( \Gamma := \partial\Omega \).

The Eulerian derivatives
The Eulerian derivatives of a shape functional are defined as follows (cf. [9, 7, 4]):

**Definition 2.9.** The first-order Eulerian derivative or the shape gradient of a shape functional \( J : \Omega \to \mathbb{R} \) at the domain \( \Omega \) in the direction of the deformation field \( V \) is given by

\[
dJ(\Omega; V) := \lim_{t \to 0^+} \frac{J(\Omega_t) - J(\Omega)}{t}, \tag{17}
\]

if the limit exists.

**Definition 2.10.** The second-order Eulerian derivative or the shape Hessian of \( J \) at the domain \( \Omega \) in the direction of the deformation fields \( V \) and \( W \) is given by

\[
d^2J(\Omega; V, W) = \lim_{s \to 0^+} \frac{dJ(\Omega_s(W); V) - dJ(\Omega; V)}{s} \tag{18}
\]

if the limit exists. Here \( \Omega_s(W) \) is the perturbed domain \( \Omega \) in the direction \( W \).

\( J \) is said to be shape differentiable at \( \Omega \) if \( dJ(\Omega; V) \) exists for all \( V \) and is linear and continuous with respect to \( V \). It is twice shape differentiable if for all \( V \) and \( W \), \( d^2J(\Omega; V, W) \) exists and if \( d^2J(\Omega; V, W) \) is bilinear and continuous with respect to \( V \) and \( W \).

3 Main Results
Here are the main results of this paper.

**Theorem 3.1.** The shape gradient of the cost functional

\[
J(\Omega) = \frac{1}{2} \int_\Omega |\nabla (u_D - u_N)|^2 \, dx
\]
in the direction of the perturbation field \( \mathbf{V} \in \Theta \), where the state functions \( u_D \) and \( u_N \) satisfy (2), and (3), respectively, is given by

\[
\frac{dJ}{d\Omega} (\Omega; \mathbf{V}) = \frac{1}{2} \int_{\Sigma} (\lambda^2 - (\nabla u_D \cdot \mathbf{n})^2 + 2\lambda\kappa u_N - (\nabla u_N \cdot \mathbf{\tau})^2) \mathbf{V} \cdot \mathbf{n} \, ds \\
+ \frac{1}{2} \int_{\Sigma} (3\alpha^2 u_N^2 - 4\alpha\lambda u_N) \mathbf{V} \cdot \mathbf{n} \, ds + \frac{1}{2} \int_{\Sigma} -2\alpha u_N u'_N \, ds.
\]

(19)

i. If \( \alpha = 0 \), then the shape gradient of the cost functional reduces to

\[
\frac{dJ}{d\Omega} (\Omega; \mathbf{V}) = \frac{1}{2} \int_{\Sigma} (\lambda^2 - (\nabla u_D \cdot \mathbf{n})^2 + 2\lambda\kappa u_N - (\nabla u_N \cdot \mathbf{\tau})^2) \mathbf{V} \cdot \mathbf{n} \, ds.
\]

(20)

ii. If \( \alpha = \kappa \), the mean curvature of \( \Sigma \), then the shape derivative becomes:

\[
\frac{dJ}{d\Omega} (\Omega; \mathbf{V}) = \frac{1}{2} \int_{\Sigma} (\lambda^2 - (\nabla u_D \cdot \mathbf{n})^2 - (\nabla u_N \cdot \mathbf{\tau})^2) \mathbf{V} \cdot \mathbf{n} \, ds + \frac{1}{2} \int_{\Sigma} 3\kappa^2 u_N^2 \mathbf{V} \cdot \mathbf{n} \, ds.
\]

(21)

Proof. Using the differentiation formula (15), we get the Eulerian derivative of \( J(\Omega) \) in the direction \( \mathbf{V} \):

\[
dJ = \int_{\Omega} \nabla(u'_D - u'_N) \cdot \nabla(u_D - u_N) \, dx + \frac{1}{2} \int_{\Sigma} |\nabla(u_D - u_N)|^2 \mathbf{V} \cdot \mathbf{n} \, ds
\]

where the shape derivatives \( u'_D \) and \( u'_N \) (at \( \Omega \) in the direction \( \mathbf{V} \)) satisfy the following boundary problems:

\[
\begin{align*}
-\Delta u'_D &= 0 \quad \text{in } \Omega, \\
u'_D &= 0 \quad \text{on } \Gamma, \\
u'_D &= -\mathbf{V} \cdot \mathbf{n} \frac{\partial u_D}{\partial \mathbf{n}} \quad \text{on } \Sigma.
\end{align*}
\]

(22)

\[
\begin{align*}
-\Delta u'_N &= 0 \quad \text{in } \Omega, \\
u'_N &= 0 \quad \text{on } \Gamma, \\
\alpha u'_N + \frac{\partial u'_N}{\partial \mathbf{n}} &= \text{div}_{\Sigma}(\mathbf{V} \cdot \mathbf{n}\nabla\Sigma u_N) - \alpha(\frac{\partial u_N}{\partial \mathbf{n}} + u_N\kappa) \mathbf{V} \cdot \mathbf{n} + \kappa\lambda \mathbf{V} \cdot \mathbf{n} \quad \text{on } \Sigma.
\end{align*}
\]

(23)

Derivations for the boundary value problems (22) and (23) can be seen in [2, 9].

Now using Green’s identity, and the BVPs (22) and (23), we write \( dJ \) as \( I_1 + I_2 \)
and manipulate each integral.

\[
I_1 = \int_\Omega \nabla (u_D' - u_N') \cdot \nabla (u_D - u_N) \, dx = \int_\Omega \nabla u_D' \cdot \nabla (u_D - u_N) \, dx - \int_\Omega \nabla u_N' \cdot \nabla (u_D - u_N) \, dx
\]

\[
= \int_\Sigma u_D' \frac{\partial}{\partial n} (u_D - u_N) \, ds - \int_\Sigma \frac{\partial u_N'}{\partial n} (u_D - u_N) \, ds
\]

\[
= - \int_\Sigma \left( \left( \frac{\partial u_D}{\partial n} \right)^2 - \frac{\partial u_D}{\partial n} \frac{\partial u_N}{\partial n} \right) V \cdot n \, ds + \int_\Sigma u_N \frac{\partial u_N'}{\partial n} \, ds
\]

\[
= - \int_\Sigma \left( \left( \frac{\partial u_D}{\partial n} \right)^2 - \frac{\partial u_D}{\partial n} (\lambda - \alpha u_N) \right) V \cdot n \, ds + \int_\Sigma \text{div}_\Sigma (V \cdot n \nabla u_N) u_N \, ds
\]

\[
- \int_\Sigma [\alpha u_N (\lambda - \alpha u_N + u_N \kappa) - \lambda u_N \kappa] V \cdot n \, ds - \int_\Sigma \alpha u_N' u_N \, ds
\]

\[
I_2 = \frac{1}{2} \int_\Sigma |\nabla (u_D - u_N)|^2 V \cdot n \, ds = \frac{1}{2} \int_\Sigma (|\nabla u_D|^2 - 2 \nabla u_D \nabla u_N + |\nabla u_N|^2) V \cdot n \, ds
\]

\[
= \frac{1}{2} \int_\Sigma \left( \left( \frac{\partial u_D}{\partial n} \right)^2 - 2 \frac{\partial u_D}{\partial n} \frac{\partial u_N}{\partial n} + (\lambda^2 - 2 \alpha \lambda u_N + \alpha^2 u_N^2) + (\nabla u_N \cdot \tau)^2 \right) V \cdot n \, ds
\]

\[
= \frac{1}{2} \int_\Sigma \left( \left( \frac{\partial u_D}{\partial n} \right)^2 - 2 \frac{\partial u_D}{\partial n} (\lambda - \alpha u_N) + (\lambda^2 - 2 \alpha \lambda u_N + \alpha^2 u_N^2) + (\nabla u_N \cdot \tau)^2 \right) V \cdot n \, ds
\]

Combining \( I_1 \) and \( I_2 \) and using the fact that

\[
\int_\Sigma \text{div}_\Sigma (V \cdot n \nabla u_N) u_N \, ds = - \int_\Sigma (\nabla u_N \cdot \tau)^2 V \cdot n,
\]

we get (19).

If \( \alpha = 0 \), then we obtain (20).

If \( \alpha = \kappa \), then \( u_N' = 0 \) by using Lemma 1 in [9]. Consequently, the shape derivative becomes (21).

\[
\Box
\]

**Remark 3.2.** For \( \alpha = 0 \) our results coincide with our results given in [3]. In [3], however, we did not utilize the shape derivatives of states in obtaining the shape gradient of the functional.

**Corollary 3.3.** At a shape \( \Omega^* \) wherein the state function \( u \) solves the Bernoulli free boundary problem (that is, \( u = u_D = u_N \) on \( \Omega^* \)), the first derivative \( dJ(\Omega; V) \) vanishes.

**Proof.** At the solution of the Bernoulli problem, \( u_D = u_N = 0 \), \( \frac{\partial u_D}{\partial \tau} = 0 \), \( \frac{\partial u_N}{\partial n} = \lambda \) on \( \Sigma \). Hence, we have

\[
dJ(\Omega; V) = \frac{1}{2} \int_\Sigma (\lambda^2 - \lambda^2 + 0 - 0) V \cdot n \, ds + 0 - 0 = 0.
\]

\[
\Box
\]
We also give a result on the second order shape derivative of the functional at the solution of the Bernoulli problem.

**Theorem 3.4.** If $u_D = u_N$ where $u_D$ and $u_N$ satisfy the Dirichlet problem (2), and the Robin boundary problem (3), respectively, then the second order shape derivative $d^2 J(\Omega; V; W)$ of the cost functional defined by

$$ J(\Omega) = \frac{1}{2} \int_{\Omega} |\nabla (u_D - u_N)|^2 \, dx $$

at $\Omega$ in the directions of the perturbation fields $V$ and $W$ is given by

$$ d^2 J(\Omega; V, W) = \int_\Sigma (\lambda^2 \nabla \cdot nS(W \cdot n) + \lambda \kappa \nabla \cdot SW \cdot n) \, ds + \int_\Sigma (\kappa u_{N,W} |\nu| \nabla \cdot n + \lambda^2 \nabla \cdot SW \cdot n) \, ds $$

\[ - \int_\Sigma (\kappa u_{N,W} V \cdot n + \lambda^2 \nabla \cdot SW \cdot n) \, ds - \int_\Sigma (\kappa u_{N,W} + \lambda^2 \nabla \cdot SW \cdot n) \, ds. \tag{24} \]

Here $S$ is an operator that relates $u'_D$ and $u'_N$ as $Su'_D = \frac{\partial u'_D}{\partial n}$, where $u'_D$ satisfies (22), $u'_N$ is the shape derivative of $u_N$ at $\Omega$ in the direction $V$ and $u'_{N,W}$ is the shape derivative of $u_N$ at $\Omega$ in the direction $W$.

i. If $\alpha = 0$, then the second order shape derivative is given by

$$ d^2 J(\Omega; V, W) = \int_{\Sigma} 2 \lambda^2 \nabla \cdot SW \cdot n \, ds + \int_{\Sigma} (S(W \cdot n) + \kappa S^{-1}(\kappa W \cdot n)) \lambda^2 \nabla \cdot SW \cdot n. $$

ii. If $\alpha = \kappa$, then the second order shape derivative of the cost functional is given by

$$ d^2 J(\Omega; V, W) = \int_{\Sigma} \lambda^2 \nabla \cdot SW \cdot n \, ds. $$

**Proof.** Let us decompose $dJ(\Omega; V)$ in Theorem 3.1 as $dJ(\Omega; V) = L + M + N$. As what we did previously, we write $L$ as $L = L_1 + L_2 + L_3$, where

$$ L_1 = \frac{1}{2} \int_\Sigma \left( \lambda^2 - \left( \frac{\partial u_D}{\partial n} \right)^2 \right) \nabla \cdot SW \cdot n \, ds, \quad L_2 = \int_\Sigma \lambda \kappa u_N \nabla \cdot SW \cdot n \, ds, $$

$$ L_3 = -\frac{1}{2} \int_\Sigma (\nabla u_N \cdot \tau)^2 \nabla \cdot SW \cdot n \, ds $$

Consider another deformation field $W$. Analogous to the previous computation, we obtain the following at the solution of the Bernoulli problem.

$$ dL_1(\Omega; W) = \int_{\Sigma} \lambda^2 (V \cdot n, (S + \kappa I)W \cdot n) \, ds, $$

$$ dL_2(\Omega; W) = \int_{\Sigma} (u'_{N,W} + \lambda W \cdot n) \lambda \kappa W \cdot n \, ds, \quad dL_3(\Omega; W) = 0, $$

where $Su'_D = \frac{\partial u'_D}{\partial n}$, and $u'_D$ satisfies (22). Therefore at the solution,

$$ dL(\Omega; W) = \int_{\Sigma} \lambda^2 (V \cdot n, (S + \kappa I)W \cdot n) \, ds + \int_{\Sigma} (u'_{N,W} + \lambda W \cdot n) \lambda \kappa W \cdot n \, ds. $$
Next we consider $M$ and derive its shape gradient at $\Omega$ in the direction $W$.

$$
M = \frac{1}{2} \int_\Sigma (3\alpha^2 u_N^2 - 4\alpha \lambda u_N) V \cdot n \, ds.
$$

$$
dM(\Omega; W) = \frac{1}{2} \int_\Sigma \left[ 6\alpha^2 u_N \cdot u_{\Gamma,N,W} - 4\alpha \lambda u_{\Gamma,N,W} \right] V \cdot n
$$

$$
+ \frac{1}{2} \int_\Sigma \left\{ \frac{\partial}{\partial n} \left[ (3\alpha^2 u_N^2 - 4\alpha \lambda u_N) V \cdot n \right] + (3\alpha^2 u_N^2 - 4\alpha \lambda u_N) V \cdot \kappa \right\} W \cdot n.
$$

At the solution of the Bernoulli problem,

$$
dM(\Omega; W) = -2 \int_\Sigma \alpha \lambda (u_{\Gamma,N,W} V \cdot n + \lambda W \cdot n V \cdot n) \, ds.
$$

Last but not least, we consider $N$ and derive also its shape gradient in the direction $W$.

$$
N = \frac{1}{2} \int_\Sigma -2\alpha u_N u'_N \, ds.
$$

$$
dN(\Omega; W) = -\int_\Sigma \left[ (\alpha u_N u'_N)'_W + \left( \frac{\partial}{\partial n} (\alpha u_N u'_N) + \alpha u_N u_{\Gamma,N,W} \right) \right] W \cdot n
$$

$$
= -\int_\Sigma \left[ \alpha u_{\Gamma,N,W} u'_N + \alpha u_N (u'_N)'_W + \left( \alpha \frac{\partial u_N}{\partial n} u'_N + \alpha u_N \frac{\partial u'_N}{\partial n} + \alpha u_N u_{\Gamma,N,W} \right) \right] W \cdot n.
$$

where $(u'_N)_W$ is the second order shape derivative of the solution $u_N$, first in the direction of the perturbation field $V$, then in the direction of the perturbation field $W$.

At the solution of the Bernoulli problem,

$$
dN(\Omega; W) = -\int_\Sigma [\alpha u_{\Gamma,N,W} u'_N + \alpha \lambda u'_N W \cdot n] \, ds.
$$

Combining $dL(\Omega; W), dM(\Omega; W)$, and $dN(\Omega; W)$, we get (24).

Now, we consider the case $\alpha = 0$. Generally, $u'_N$ satisfies the variational equation:

$$
\int_\Sigma \left( \frac{\partial u'_N}{\partial n} + \alpha u'_N \right) \varphi = \int_\Sigma -\nabla_N u_N \nabla \varphi V \cdot n - \alpha \left( \frac{\partial u_N}{\partial n} + u_N \kappa \right) \varphi V \cdot n + \lambda \kappa \varphi V \cdot n.
$$

where $\varphi \in H^1(\Omega; \Gamma)$. For this case, at the solution of the Bernoulli problem, $u'_N$ satisfies the following reduced variational equation:

$$
\int_\Sigma \left( \frac{\partial u'_N}{\partial n} - \lambda \kappa V \cdot n \right) \varphi = 0
$$

And by the fundamental lemma of calculus of variations, we get

$$
\frac{\partial u'_N}{\partial n} - \lambda \kappa V \cdot n = 0
$$
or equivalently, \( \frac{\partial u'_N}{\partial n} = \lambda \kappa \mathbf{V} \cdot \mathbf{n} \). Using the Steklov-Poincare operator: \( Su'_N = \frac{\partial u'_N}{\partial n} \), we obtain

\[
   u'_N = S^{-1}(\lambda \kappa \mathbf{V} \cdot \mathbf{n}) \tag{25}
\]

Consequently,

\[
   u'_{N,W} = S^{-1}(\lambda \kappa \mathbf{W} \cdot \mathbf{n}). \tag{26}
\]

Substituting \( \alpha = 0 \), (25), and (26) into (24), we get

\[
d^2 J(\Omega; \mathbf{V}, \mathbf{W}) = \int_{\Sigma} 2\lambda^2 \kappa \mathbf{V} \cdot \mathbf{n} \mathbf{W} \cdot \mathbf{n} \, ds + \int_{\Sigma} (S(\mathbf{W} \cdot \mathbf{n}) + \kappa S^{-1}(\kappa \mathbf{W} \cdot \mathbf{n}))\lambda^2 \mathbf{V} \cdot \mathbf{n}. \]

For \( \alpha = \kappa \), we note that \( u'_N = 0 \) and \( u'_{N,W} = 0 \) by applying Lemma 1 of [9]. Hence, we obtain

\[
d^2 J(\Omega; \mathbf{V}, \mathbf{W}) = \int_{\Sigma} \lambda^2 \mathbf{V} \cdot \mathbf{n} S(\mathbf{W} \cdot \mathbf{n}) \, ds. \]

Remark 3.5. For \( \alpha = 0 \), our results coincides with the one presented in [1] wherein three strategies were utilized to derive the shape Hessian of the functional.

Acknowledgement

The work was supported by University of the Philippines System under the award Diamond Jubilee Professorial Chair. The results were presented in the 2014 Taiwan-Philippines Symposium on Analysis, which was held at Kaohsiung, Taiwan on March 31 - April 3, 2014 through the travel support of the conference organizers.

References


Received: July 29, 2014