On the Parametrical Lattice Boltzmann Equations

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Abstract

Lattice Boltzmann equations with dependence on a parameter are proposed. The construction of finite-difference schemes is based on the integral form of the system of kinetic equations with discrete velocities. The schemes are developed with usage of quadrature formulas. It is demonstrated, that at some parameter values the second order of approximation takes place. The stability conditions are obtained. The possibilities of application to fluid dynamics problems are demonstrated on the solutions of 2D lid-driven cavity flow problem and problem of the flow in channel with rectangular stricture. Obtained results are compared with the results obtained by other numerical methods.

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1 Introduction

The lattice Boltzmann method (LBM) is a powerful technique for the computations of a wide variety of complex fluid dynamics problems, such as single and multiphase fluid flows in complex geometries [3],[9]. In the last years, LBM proved to be a viable alternative to traditional numerical methods, based on solution of Navier–Stokes equations. Among the reasons of its increasing popularity are probably its simplicity, implementation easiness and parallelism of its algorithm.
According to LBM, the fluid is modelled as an ensemble of particles, which can perform collision and streaming processes on a discrete lattice mesh. In this approach, a Bhatnagar–Gross–Krook (BGK) kinetic equation \cite{2} is discretized in velocity space in such a way that the Navier–Stokes equations are obtained in the macroscopic limit by Chapman–Enskog expansion method. Due to its particular nature and local dynamics, LBM has some advantages over the traditional numerical methods. From a computational viewpoint, these advantages are parallelism of the algorithm and ease of incorporating microscopic interactions. In last years the increasing of popularity of LBM can be explained by the fact, that the method is successfully adopted for computations on single and multiple graphical processing units (GPU) using Compute Unified Device Architecture (CUDA) technology \cite{11}, \cite{13}.

A disadvantage of the LBM adoption for realistic flows is its conditional numerical stability due to the explicit nature of its numerical scheme. This disadvantage plays important role for the cases of flows with small viscosity, such as turbulent flows.

In this study, the parametrical lattice Boltzmann equations (LBE’s) for the stabilization of LBM by variation of parameter values are constructed. Construction of parametrical LBE’s are performed by integro-interpolative method \cite{7} application to the integral form of BGK equations with discrete velocities. Simple stability analysis of the constructed LBE’s is performed. The validity of LBE’s application to practical problems is demonstrated by solutions of two test problems.

The paper is organized as follows. In Section 2 the LBE is discussed. In Section 3, parametrical LBE’s are constructed. In Section 4, simple stability analysis is performed. In Section 5, two test problems are considered and numerical results are discussed. Concluding remarks are made in Section 6.

2 Lattice Boltzmann equation

In LBM the fluid is represented by an ensemble of mesoscopic particles that move on regular spatial lattice and undergo collisions at its nodes. Velocities of the particles are formed a discrete set of vectors $V_i = Vv_i, i = 1, n, n \in \mathbb{N}$, where $V = l/\delta t$ is a typical velocity, $\delta t$ is a time step, $l$ is a lattice spacing (spatial step). In this paper the case of the absence of the body force is considered.

The particle distribution functions $f_i$ are used as main variables. The evolution of $f_i$ are described by the system of BGK kinetic equations with discrete velocities $V_i$:

$$\frac{\partial f_i}{\partial t} + V_{ix} \frac{\partial f_i}{\partial x} + V_{iy} \frac{\partial f_i}{\partial y} = -\frac{1}{\lambda} \left( f_i - f_i^{(eq)} \right),$$

(1)
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where $\lambda$ is a relaxation time, $f_i^{(eq)}$ are the equilibrium distribution functions.

The fluid density $\rho$ and the velocity $u$ at lattice node $r_{kl} = (x_k, y_l)$ are calculated as:

$$
\rho(t, r_{kl}) = \sum_{i=1}^{n} f_i(t, r_{kl}), \quad \rho(t, r_{kl}) u(t, r_{kl}) = \sum_{i=1}^{n} V_i f_i(t, r_{kl}). \tag{2}
$$

In applications of LBM to the incompressible viscous fluid flows modelling, the case of small values of Mach number $M = |u|/V$ is considered. In this case the expressions for $f_i^{(eq)}$ can be written as [3]:

$$
f_i^{(eq)} = W_i \rho \left( 1 + 3 \frac{(V_i \cdot u)}{V^2} + \frac{9}{2} \frac{(V_i \cdot u)^2}{V^4} - \frac{3}{2} \frac{u^2}{V^2} \right), \tag{3}
$$

where $W_i = 4/9$ for $i = 1$; $1/9$, for $i = 2, 3, 4, 5$; $1/36$, for $i = 6, 7, 8, 9$.

Numerical schemes for the solution of (1) can be constructed with the usage of integral form of (1), which can be written as [4], [8], [9]:

$$
f_i(t + \delta t, r + V_i \delta t) - f_i(t, r) = - \frac{1}{\lambda} \int_0^\delta t (f_i(t + \xi, r + V_i \xi) - f_i^{(eq)}(t + \xi, r + V_i \xi)) d\xi. \tag{4}
$$

System (4) is constructed by the integration of (1) along characteristics in phase space.

Finite-difference schemes for LBM can be constructed from (4) by application of integro-interpolative method [7] with usage of quadrature formulas for the computation of integral in (4).

By the application of left point quadrature formulae the following scheme can be constructed:

$$
f_i(t_j + \delta t, r_{kl} + V_i \delta t) - f_i(t_j, r_{kl}) = - \frac{1}{\tau} \left( f_i(t_j, r_{kl}) - f_i^{(eq)}(t_j, r_{kl}) \right), \tag{5}
$$

where $t_j$ is a node of the time grid constructed with step $\delta t$, $\tau = \lambda / \delta t$ is a dimensionless relaxation time. Scheme (5) is also known as lattice Boltzmann equation.

The expression for viscosity coefficient $\nu$ for the scheme (5) can be obtained by the Chapman – Enskog expansion method. The expression for the D2Q9 lattice, which is applied to planar flows computation and is characterized by vectors:

$$
\mathbf{v}_1 = (0, 0), \quad \mathbf{v}_2 = (1, 0), \quad \mathbf{v}_3 = (0, 1), \quad \mathbf{v}_4 = (-1, 0), \quad \mathbf{v}_5 = (0, -1),
$$

$$
\mathbf{v}_6 = (1, 1), \quad \mathbf{v}_7 = (-1, 1), \quad \mathbf{v}_8 = (-1, -1), \quad \mathbf{v}_9 = (1, -1),
$$

has the following form:

$$
\nu = \left( \frac{\tau - 1}{2} \right) \frac{l^2}{3\delta t}. \tag{6}
$$

Due to (6), the stability condition for (5) can be expressed as $\tau > 1/2$. 
3 Parametrical lattice Boltzmann equations

The construction of parametrical LBE’s is performed with the usage of Ashyraliev’s approach [1]. Let us add and deduct in the right part of (4) the following term:

\[ \frac{\sigma}{\lambda} \int_0^{\delta t} (f_i(t + \xi, r + V_i \xi) - f_i^{(eq)}(t + \xi, r + V_i \xi)) d\xi, \]

where \( \sigma \in [0,1] \) is a dimensionless parameter. After these operations we obtain:

\[ f_i(t + \delta t, r + V_i \delta t) - f_i(t, r) = -\frac{1 - \sigma}{\lambda} \int_0^{\delta t} (f_i(t + \xi, r + V_i \xi) - f_i^{(eq)}(t + \xi, r + V_i \xi)) d\xi, \]  

(7)

In this study the following quadrature formulas are used for the computations of integrals in (6):

1) Left point rule:

\[ b \int_a F(x) dx \approx F(a)(b - a). \]  

(7.1)

2) Right point rule:

\[ b \int_a F(x) dx \approx F(b)(b - a). \]  

(7.2)

3) Trapezoidal rule:

\[ b \int_a F(x) dx \approx \frac{b - a}{2}(F(a) + F(b)). \]  

(7.3)

Six systems of parametrical LBE’s were constructed. These systems can be represented by the following formulae:

\[ f_i(t_j + \delta t, r_{kl} + V_i \delta t) - f_i(t_j, r_{kl}) = A(f_i(t_j, r_{kl}) - f_i^{(eq)}(t_j, r_{kl})) + \\
+ B(f_i(t_j + \delta t, r_{kl} + V_i \delta t) - f_i^{(eq)}(t_j + \delta t, r_{kl} + V_i \delta t)). \]  

(8)

The dependencies of \( A \) and \( B \) on \( \sigma \) are defined the following systems of LBE’s:


\[ A = -\frac{1 - \sigma}{\tau}, \quad B = -\frac{\sigma}{\tau}, \]  

(8.1)
these expressions for $A$ and $B$ are obtained from (7) with the approximation of first term in right part by (7.1) and with the approximation of second term by (7.2).

2. System 2.

$$A = -\frac{\sigma}{\tau}, \quad B = -\frac{1 - \sigma}{\tau},$$

(8.2)

expressions for these coefficients are obtained when first term in right part of (6) is approximated by (7.2) and the second one is approximated by formulae (7.1).


$$A = \frac{1 - \sigma}{2\tau}, \quad B = \frac{1 + \sigma}{2\tau},$$

(8.3)

these expressions are obtained when first term in right part of (6) is approximated by (7.3), and second one — by (7.1).


$$A = -\frac{1 + \sigma}{2\tau}, \quad B = -\frac{1 - \sigma}{2\tau},$$

(8.4)

these expressions are obtained when first term in right part of (6) is approximated by (7.3), and second one — by (7.2).

5. System 5.

$$A = -\frac{\sigma}{2\tau}, \quad B = \frac{\sigma - 2}{2\tau},$$

(8.5)

these expressions are obtained when first term in right part of (6) is approximated by (7.1), and second one — by (7.3).


$$A = \frac{\sigma - 2}{2\tau}, \quad B = -\frac{\sigma}{2\tau},$$

(8.6)

these expressions are obtained when first term in right part of (6) is approximated by (7.2), and second one — by (7.3).

It is easy to obtain that from systems 1–6 the scheme with second order of approximation can be constructed. This scheme corresponds to $\sigma = 1/2$ for (8.1) and (8.2), to $\sigma = 0$ for (8.3) and (8.4) and to $\sigma = 1$ for (8.5) and (8.6). For other values of $\sigma$ only first order of approximation can be obtained. The scheme of second order has implicit nature according to dependence of (2) and (3) on $f_i$. In practical realization of this scheme there is a need to solve system of nonlinear algebraic equations in every lattice node.

The expression for viscosity $\nu$ in case of D2Q9 lattice is obtained by Chapman – Enskog expansion method application to (6) and can be written as:

$$\nu = \left(1 + \frac{A + B}{2}\right) \frac{\tau l^2}{3\delta t}.$$  

(9)
The expression for $\nu$ in case of second order scheme has the following form:

$$\nu = \frac{\tau l^2}{3\delta t}.$$ 

This expression is coincide with the expression of $\nu$ for the system (1) [10].

4 Stability analysis

Stability investigation for the systems represented by formulae (8) is performed in the case of the absence of perturbations in coordinates for distribution functions $f_i$. This case corresponds to simple dependencies: $f_i = f_i(t)$.

Let us consider the solutions of (8) in this case in the following form:

$$f_i(t) = \bar{f}_i + \delta f_i(t), \quad (10)$$

where $\bar{f}_i$ formed the vector of unperturbed solution of (8), which correspond to initial conditions $f_i(0) = f_i^{(eq)}(\tilde{\rho}, \tilde{u})$, where $\tilde{\rho}$ is a constant and $\tilde{u}$ is a constant vector, $\delta f_i(t)$ are the perturbations.

After the substitution of (10) into (8) the following system for $\delta f_i(t)$ is obtained:

$$\delta f_i(t_j + \delta t) = \frac{1 + A}{1 - B} \delta f_i(t_j). \quad (11)$$

Stability condition of the null solution of (11) is expressed by inequality:

$$\left|\frac{1 + A}{1 - B}\right| < 1.$$ 

For systems 1–6 this inequality is followed by following stability conditions:

1. For system 1: $\tau + \sigma > 1/2$, since $\sigma \in [0, 1]$ and $\tau$ must be positive, so in the case of $\sigma \in [1/2, 1]$ system is stable for all $\tau$ values.

2. For system 2: $\tau - \sigma > -1/2$, and system is stable in case of $\sigma \in [0, 1/2]$ for all values of $\tau$.

3. For system 3: $\tau > \sigma/2$, and system is stable for all values of $\sigma$ and $\tau$.

4. For system 4: $\tau > -\sigma/2$, and system is stable for all values of $\sigma$ and $\tau$.

5. For system 5: $\tau > (1 - \sigma)/2$, and system is stable for all values of $\tau$ only in the case of $\sigma = 1$.

6. For system 6: $\tau > (\sigma - 1)/2$, and system is stable for all values of $\sigma$ and $\tau$.

The scheme of the second order is stable for all values of $\tau$. 
5 Solution of test problems

The application of the developed second order implicit scheme to practical fluid dynamics problems is demonstrated on the solution of two test problems: the 2D lid-driven cavity flow problem and the problem of the flow in channel with rectangular stricture. Obtained numerical results are compared with results obtained by LBE (5) for both problems. Results for 2D lid-driven cavity flow problem are compared with results presented in [6], obtained by finite-difference method. Results for the flow in channel with rectangular stricture are compared with solution obtained by finite-element method realized in GNU licensed software freem++ v.3.20 [5].

The system of nonlinear equations (8) in every lattice node is solved by Newton – Raphson’s method. The convergence of the method is regulated by the norm of the difference of the solutions on two adjacent iterations. Iteration process is stopped when the norm is less than $10^{-8}$. It is demonstrated that maximal number of iterations is equal to 35.

The main input parameter of the program is Reynolds number $Re$, which is calculated as:

$$Re = \frac{US}{\nu},$$

(12)

where $S$ is a typical length, $U$ is a typical velocity. Parameters $U$ and $S$ are input before the computation. Viscosity coefficient $\nu$ is calculated by (12) from the values of $Re$, $S$ and $U$. Parameter $\tau$ is calculated from the value of $\nu$ by (9).

The values of $f_i$ at the initial time moment $t = 0$ are chosen to be equal to the values of $f_i^{(eq)}$. The velocity $u$ at this moment is chosen as zero vector at all lattice nodes and the initial density is chosen to be equal to unity. The boundary conditions for $f_i$ are realized by Zou and He method [14].

5.1 2D lid-driven cavity flow. The problem is stated in 2D domain $\{(x, y) | x \in [0, P], y \in [0, P], P > 0\}$ (fig. 1).
The boundary conditions has a following form:

\[ u_x(t, x, 0) = u_y(t, x, 0) = 0, \quad u_x(t, x, P) = U_0, u_y(t, x, P) = 0, \quad x \in [0, P], \]

\[ u_x(t, 0, y) = u_y(t, 0, y) = u_x(t, P, y) = u_y(t, P, y) = 0, \quad y \in [0, P), \]

where \( U_0 = \text{const} \neq 0 \) is the velocity of the top lid. These conditions are called no-slip conditions on all boundaries 1–4 of computational domain. For the \( Re \) computation in (12) it can be proposed that \( U = U_0, S = P \). The following values are used: \( P = 1 \text{ m}, U_0 = 0.1 \text{ m/s} \). The time interval of 1000 s is considered.

The computations are performed on grids with \( 50 \times 50, 100 \times 100, 200 \times 200 \) and \( 500 \times 500 \) nodes with \( 10^4, 2 \cdot 10^4, 4 \cdot 10^4 \) and \( 10^5 \) time steps. Reynolds number is chosen to be equal to 100. It is demonstrated that stationary flow regime is realized after some time interval, e. g. for grid with \( 50 \times 50 \) nodes this regime is realized after \( 25 \cdot 10^3 \) time steps.

At fig. 2 the plots of dimensionless velocity components are presented for the case of the grid with \( 200 \times 200 \) nodes in the case of stationary flow regime. As it can be seen, obtained results are in good agreement with results from [6].

5.2. Flow in channel with rectangular stricture. The computational domain is presented on fig. 3. Boundaries 1 and 2 are fixed, boundaries 3 and 4 are inflow and outflow boundaries of the domain. The boundary conditions are no-slip conditions on boundaries 1 and 2:

\[ u_x(t, x, 0) = u_y(t, x, 0) = u_x(t, x, H) = u_y(t, x, H) = 0, \]

and on boundaries 3 and 4 velocity components are defined as:

\[ u_x(t, 0, y) = u_x(t, L, y) = -4 \frac{U_0}{H^2} y (y - H), \quad u_y(t, 0, y) = u_y(t, L, y) = 0. \]
Fig. 2. Solutions of the 2D lid-driven cavity flow problem. A — plot of $u_x/U$, B — plot of $u_y/U$. 1 — result from [6]; 2 — result obtained by LBE (5); 3 — result obtained by second order LBE.

The values of the parameters are: $L = 2$ m, $H = 0.5L$, $U_0 = 0.1$ m/s. The value of $Re$ is equal to 200, value of $S$ from (12) is equal to $0.2H$, $U$ is equal to $U_0$. The grid with $400 \times 400$ nodes is considered. Time interval is equal to 1000 s, the number of time steps is equal to $2 \cdot 10^4$. Stationary regime is realized after $15 \cdot 10^3$ time steps.

Obtained results are compared with results obtained after the solution of Navier — Stokes equations by Chorin — Rannacher’s splitting method with
finite elements [10]. Program realization of the algorithm of this method is performed in freefem++ v. 3.20 [5]. The time derivative is approximated by forward finite difference, the same number of time steps as for LBE is chosen. The unstructured mesh of triangle elements with 9175 nodes is used. Linear interpolation functions are considered.

Fig. 3. Computational domain for the problem of the flow in channel with rectangular stricture

![Computational domain](image)

Fig. 4. Solutions of the problem of the flow in channel with rectangular stricture.

Plot of $u_x$ in case of $x = 0.5L$, $y \in [0, H]$. 1 — result obtained by Chorin — Rannacher’s method; 2 — result obtained by LBE (5); 3 — result obtained by second order LBE.

The obtained results are demonstrated at fig. 4. As it can be seen, results are in good agreement with results obtained by Chorin — Rannacher’s method.
6 Summary

In the article parametrical LBE’s are constructed with the usage of integro-interpolative method. The stability of parametrical LBE’s can be regulated by variation of parameter values. Stability conditions are obtained. Possibilities of parametrical LBE’s application to practical problems are demonstrated on the solutions of two test problems. Obtained results showed a good agreement with results obtained by other numerical methods.

The proposed method of construction of parametrical LBE’s can be applied for construction of multistep LBM schemes and for LBE’s with a few parameters.

References


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