On the Upper Chromatic Number
of Uniform Hypergraphs

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Abstract

In this paper we determine some necessary conditions for a uniform hypergraph to have a given upper chromatic number.

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1 Introduction

A hypergraph $\mathcal{H}$ of order $n$ is the pair $(X, \mathcal{B})$, where $X$ is a vertex set, with $|X| = n$, and $\mathcal{B}$ a family of nonempty subsets of $X$, called edges, such that $\bigcup_{E \in \mathcal{B}} E = X$. Some edges may coincide as subsets; they are then called repeated. The number of distinct edges in $\mathcal{B}$ is denoted by $m_{\mathcal{H}}$, i.e., $m_{\mathcal{H}}$ does not count repetitions of edges. If all the edges of $\mathcal{B}$ have the same cardinality $\varrho$, then $\mathcal{H}$ is called uniform hypergraph of rank $\varrho$ and the elements of $\mathcal{B}$ are also called blocks. When all the edges are distinct, $\mathcal{H}$ is called simple.
1) A C-colouring of \( \mathcal{H} \) is a surjective mapping \( f: X \rightarrow C \) such that in every edge \( E \in \mathcal{B} \) there are at least two distinct vertices having the same colour. The maximum number of colours for which there exists a C-colouring of \( \mathcal{H} \) is called the upper chromatic number of \( \mathcal{H} \) and it is denoted by \( \chi_{\mathcal{H}} \). Evidently, the minimum number of colours that can be used in a C-colouring is 1. A hypergraph \( \mathcal{H} \) with a C-colouring is called a C-hypergraph.

2) A D-colouring of \( \mathcal{H} \) is a surjective mapping \( f: X \rightarrow C \) such that in every edge \( E \in \mathcal{B} \) there are at least two distinct vertices having different colours. The minimum number of colours for which there exists a D-colouring of \( \mathcal{H} \) is called the lower chromatic number of \( \mathcal{H} \) and it is denoted by \( \chi_{\mathcal{H}} \). Evidently, the maximum number of colours that can be used in a D-colouring is \( n \). A hypergraph \( \mathcal{H} \) with a D-colouring is called a D-hypergraph.

3) A strong colouring of \( \mathcal{H} \), or S-colouring, is a surjective mapping \( f: X \rightarrow C \) such that for every edge \( E \in \mathcal{B} \) the number of vertices in the image of \( f \) is equal to the size of \( E \). The minimum number of colours for which there exists a strong colouring of \( \mathcal{H} \) is called the strong chromatic number of \( \mathcal{H} \) and is denoted by \( \chi^*_{\mathcal{H}} \). A hypergraph \( \mathcal{H} \) with an S-colouring is called an S-hypergraph.

In each of the types of colourings, when a colouring \( f \) uses \( k \) colours, then it is said to be a \( k \)-colouring of \( \mathcal{H} \). Each \( k \)-colouring defines a partition of \( X \) into \( k \) colour-classes \( X_i \), whose cardinalities will be denoted by \( n_i \), for \( 1 \leq i \leq k \).

Some of the concepts above have been introduced in [18, 19], and many related results can be found in [14, 17], and those related to Steiner systems in [1, 15]. For the following concepts, see [11, 14].

Let \( C \) be a set of colours and let \( X \) be any set of elements. A distribution of colours of \( C \) on \( X \) [DC for short], is a surjection \( \Delta: X \rightarrow C \). If \( |C| = k \), then \( \Delta \) partitions the vertex set \( X \) into \( k \) colour-classes \( X_i \), for \( i = 1, 2, \ldots, k \), whose cardinality will be denoted by \( n_i \).

Let \( \Delta \) be a DC of \( C \) on \( X \). Let \( q \) and \( r \) be the quotient and the remainder of division of \( |X| \) by \( |C| \) respectively. If \( \Delta \) defines \( r \) colour classes containing \( q+1 \) vertices and \( |C| - r \) colour classes containing \( q \) vertices, then \( \Delta \) is said to be an equidistribution of colours of \( C \) on \( X \) [EC for short]. An equidistribution of colours will be denoted by \( E \). As one can see, in \( E \) the cardinalities of colour classes differ by at most 1.

Let \( \varrho \) be a positive integer, and let \( \Delta \) be a DC of \( C \) on \( X \). Consider the family \( \mathcal{B} \) of all possible distinct subsets of \( X \) having cardinalities \( \varrho \) [\( \varrho \)-uples] for which \( \Delta \) satisfies the conditions for the colourings of type 1), or 2), or 3). The pair \( \mathcal{H} = (X, \mathcal{B}) \) defines a uniform hypergraph of rank \( \varrho \), which is called the hypergraph associated with \( \Delta \) and it will be denoted by \( \mathcal{H}(\varrho, \Delta) \).

The hypergraph strongly associated with \( \Delta \) will be denoted by \( \mathcal{K}(\varrho, \Delta) \), and
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in case $EC$, it will be denoted by $K(\varrho, E)$. The edges of $K(\varrho, \Delta)$ are formed by all the subsets of $X$ having cardinality $\varrho$ and such that no two vertices of every edge are in the same class of $\Delta$.

In [11] the authors determine necessary conditions for a uniform hypergraph to have a given chromatic number and a given strong chromatic number.

In this paper, we determine some necessary conditions for a uniform hypergraphs to have a given upper chromatic number and study the case of bihypergraphs, i.e. hypergraphs satisfying both 1) and 2) conditions simultaneously. In what follows, given two positive integers $k$ and $h$, by $Q(k, h)$ and $R(k, h)$ we denote the quotient and the remainder of division of $k$ by $h$, that is $k = Q(k, h)h + R(k, h)$.

2 Combinatorial formulas and some results

In this section we provide some properties of distributions and equidistributions of colours. The first two lemmas are immediate by pigeon-hole principle. The third lemma is a technical result of combinatorial calculus. We refer the reader to [11] for all the proofs.

**Lemma 2.1** Let $\Delta : X \rightarrow C$ be a colour distribution, $|X| = n$, $|C| = p$, $q = Q(n, p)$, and $r = R(n, p)$. Then the following implications hold:

i) If $\Delta$ has a colour-class $X_i$ with $n_i < q$ vertices, then there exists a colour-class $X_j$ with $n_j \geq q + 1$ vertices.

ii) If $\Delta$ has a colour-class $X_j$ with $n_j > q + 1$ vertices, then there exists a colour-class $X_i$ with $n_i \leq q$ vertices.

**Lemma 2.2** A distribution of colours $\Delta$ is an equidistribution of colours if and only if $|X_i - X_j| \leq 1$, for every pair of colour-classes $X_i$, $X_j$.

**Lemma 2.3** If $\varrho$, $s$, $t \in \mathbb{N}$ and $1 < \varrho \leq s < t - 1$, then

$$\binom{t}{\varrho} - \binom{t - 1}{\varrho} > \binom{s + 1}{\varrho} - \binom{s}{\varrho}.$$

**Lemma 2.4** Let $\Delta$ be a DC of $C$ on $X$ which defines two colour-classes $X_i$ and $X_j$ such that $n_j > n_i + 1$. If $\Delta'$ is a DC obtained by moving a vertex from $X_j$ to $X_i$, then

$$m_H(e, \Delta) < m_H(e, \Delta').$$
Theorem 2.1 If $\Delta$ and $E$ are respectively a DC and an EC of $C$ on $X$, then:

$$m_{\mathcal{H}(\varrho,\Delta)} \leq m_{\mathcal{H}(\varrho,E)}.$$ 

Corollary 2.1 If $\Delta$ is a distribution of colours of $C$ on $X$, then $\mathcal{H}(\varrho,\Delta)$ has the maximum number of blocks if and only if $\Delta$ is an equidistribution of colours.

Theorem 2.2 Let $E$ be an equidistribution of colours of $C$ on $X$, with $|X| = n$ and $|C| = p$. If $q = Q(n,p)$ and $r = R(n,p)$, then

$$m_{\mathcal{H}(\varrho,E)} = \binom{n}{q} - \binom{q+1}{q}r - \binom{q}{q}(p-r).$$

Theorem 2.3 If $E$, $E'$ are two equidistributions of colours respectively of $C$ and of $C'$ on $X$, with $|X| = n$, $|C| = p$ and $|C'| = p+1$, then

$$m_{\mathcal{H}(\varrho,E)} \leq m_{\mathcal{H}(\varrho,E')}.$$ 

Lemma 2.5 Let $\Delta$ be a distribution of colours of $C$ on $X$ having two colour classes $X_i$ and $X_j$, with $n_j > n_i + 1$. If $\Delta'$ is a distribution of colours obtained from $\Delta$ by moving a vertex from $X_j$ to $X_i$, then

$$m_{\mathcal{K}(\varrho,\Delta)} < m_{\mathcal{K}(\varrho,\Delta')}.$$ 

Theorem 2.4 If $\Delta$ and $E$ are respectively a distribution of colours and an equidistribution of colours of $C$ on $X$, then

$$m_{\mathcal{K}(\varrho,\Delta)} \leq m_{\mathcal{K}(\varrho,E)}.$$ 

Corollary 2.2 If $\Delta$ is a distribution of colours of $C$ on $X$, then $\mathcal{K}(\varrho,\Delta)$ has the maximum number of edges if and only if $\Delta$ is an equidistribution of colours.

Theorem 2.5 Let $E$ be an equidistribution of colours of $C$ on $X$, with $|C| = p$ and $|X| = n$. If $q = Q(n,p)$ and $r = R(n,p)$, then

$$m_{\mathcal{K}(\varrho,E)} = \sum_{i=0}^{e} \binom{r}{i} \binom{p-r}{q-i} (q+1)^{e-i}.$$ 

Theorem 2.6 If $E$ and $E'$ are two equidistributions of colours of $C$ on $X$, with $|E| = p$ and $|E'| = p+1$, then

$$m_{\mathcal{K}(\varrho,E)} \leq m_{\mathcal{K}(\varrho,E')}.$$
3 D-Hypergraphs and S-Hypergraphs

Consequences of these combinatorial results are the following necessary conditions for D-hypergraphs and S-hypergraphs proved in [11]:

**Theorem 3.1** Let $\mathcal{H}=(X,B)$ be a uniform hypergraph of rank $\varrho$ and order $n$, having chromatic number $\chi_H = \chi$. If $q = Q(n, \chi)$ and $r = R(n, \chi)$, then

$$m_H \leq \binom{n}{\varrho} - \binom{q + 1}{\varrho} r - \binom{q}{\varrho} (\chi - r).$$

**Theorem 3.2** Let $\mathcal{H}=(X,B)$ be a uniform hypergraph of rank $\varrho$. If $q = Q(n, \chi^*)$ and $r = R(n, \chi^*)$, where $\chi^* = \chi_H^*$, then

$$m_H \leq \sum_{i=0}^{\varrho} \binom{r}{i} \left( \binom{\chi^* - r}{\varrho - i} (q + 1)^i q^{e-i} \right).$$

4 V-Hypergraphs

Given $\varrho \in \mathbb{N}$ and $\Delta$ as a DC of a colouring $\mathcal{C}$ on $X$, it is possible to define the V-hypergraph associated with $\Delta$ and denoted by $\mathcal{V}(\varrho, \Delta)$. In the case of $EC$ the respective hypergraph is denoted by $\mathcal{V}(\varrho, E)$. The edges of $\mathcal{V}(\varrho, \Delta)$ are formed by all the subsets of $X$ of cardinality $\varrho$ and having at least two vertices in the same class of $\Delta$.

Observe that, since a $\mathcal{V}$-colouring defined in a hypergraph $H = (X,B)$ is a $\mathcal{C}$-colouring of $H$, all the results about $\mathcal{C}$-hypergraphs from Section 2 are also valid for $\mathcal{V}$-hypergraphs.

Consider $\varrho \in \mathbb{N}$ and $\Delta$ a DC of $\mathcal{C}$ on $X$, where $|X| = n$. If the classes of $\Delta$ have all cardinality one, except at most one class, then we say that $\Delta$ is an antiequidistribution of colours of $\mathcal{C}$ on $X$, briefly an AD.

**Theorem 4.1** Let AD be an antiequidistribution of colours of $\mathcal{C}$ on $X$, with $|X| = n$ and $|\mathcal{C}| = p$. If $n \geq p + 1$, then

$$m_{\mathcal{V}(\varrho, \mathcal{A})} = \sum_{i=2}^{\varrho} \binom{n - p + 1}{i} \cdot \binom{p - 1}{\varrho - i}.$$ 

**Proof.** Let $C_1, C_1, ..., C_p$ be the colour classes of $\mathcal{C}$, with $|C_1| = n - p + 1$ and $|C_2| = |C_3| = ... = |C_{p-1}| = 1$. To prove the statement, it is necessary to calculate the number of all the $\varrho$-tuples having at least two elements in $C_1$. Therefore,

$$m_{\mathcal{V}(\varrho, \mathcal{A})} = \binom{n - p + 1}{2} \binom{p - 1}{\varrho - 2} + \binom{n - p + 1}{3} \binom{p - 1}{\varrho - 3} + .... +$$
\begin{equation}
\binom{n-p+1}{\varrho-1}(p-1) + \binom{n-p+1}{\varrho}.
\end{equation}

\begin{proof}
Theorem 4.2 Let \( \Delta \) be a DC of \( C \) on \( X \) which defines two colour-classes \( X_i \) and \( X_j \) such that \( n_j > n_i + 1 \). If \( \Delta' \) is a DC obtained by moving a vertex from \( X_i \) to \( X_j \), then

\begin{equation}
m_{V(\varrho,\Delta)} < m_{V(\varrho,\Delta')}.
\end{equation}

\end{proof}

\begin{proof}
Observe that \( m_{V(\varrho,\Delta)} \) can be calculated as follows:

\begin{equation}
m_{V(\varrho,\Delta)} = \binom{n}{\varrho} - m_{K(\varrho,\Delta)}.
\end{equation}

It is obtained by considering the number of all \( \varrho \)-tuples on \( n \) vertices minus the number of \( \varrho \)-tuples having all different colours. Therefore:

\begin{equation}
m_{V(\varrho,\Delta')} = \binom{n}{\varrho} - m_{K(\varrho,\Delta')},
\end{equation}

and, since \( m_{K(\varrho,\Delta)} > m_{K(\varrho,\Delta')} \) by Lemma 2.5, it follows that \( m_{V(\varrho,\Delta)} < m_{V(\varrho,\Delta')} \).

\begin{proof}
Theorem 4.3 If \( \Delta \) and \( A \) are a distribution and an antiequidistribution of colours of \( C \) on \( X \) respectively, then

\begin{equation}
m_{V(\varrho,\Delta)} \leq m_{V(\varrho,A)}.
\end{equation}

\end{proof}

\begin{proof}
If \( \Delta \) is an antiequidistribution of colours, then \( m_{V(\varrho,\Delta)} = m_{V(\varrho,A)} \). If \( \Delta \) is not an AC then there exist at least two colour-classes of \( \Delta \), say \( X_i \) and \( X_j \), having cardinalities \( n_i \) and \( n_j \) respectively with \( n_j > n_i + 1 \). Move a vertex from \( X_j \) to \( X_i \) and obtain a new distribution of colours \( \Delta_1 \). Repeating the step until obtaining an antiequidistribution of colours implies:

\begin{equation}
m_{V(\varrho,\Delta)} < m_{V(\varrho,\Delta_1)} < \ldots < m_{V(\varrho,A)}.
\end{equation}

\end{proof}
Theorem 4.4 If $A$ is an antiequidistribution of colours of $C$ on $X$, with $|C| = p$ and $|X| = n$, then

$$m_{K(\varrho, A)} = \binom{p-1}{\varrho} + \binom{p-1}{\varrho-1}(n-p+1).$$

Proof. Let $C_1, C_1, ..., C_p$ be the colour classes of $C$ with $|C_1| = n - p + 1$ and $|C_2| = |C_3| = ... |C_{p-1}| = 1$. To calculate $m_{K(\varrho, A)}$ consider the number of all $\varrho$-tuples having exactly one vertex in each class. Thus, $\binom{p-1}{\varrho}$ is the number of such $\varrho$-tuples having no vertices in $C_1$, and $\binom{p-1}{\varrho-1}(n-p+1)$ is the number of all the remaining tuples. $\square$

Corollary 4.1 If $\Delta$ is a distribution of colours of $C$ on $X$, then $K(\varrho, \Delta)$ has the minimum number of edges if and only if $\Delta$ is an antiequidistribution of colours.

Proof. It follows directly from Theorem 4.3. $\square$

Theorem 4.5 If $\mathcal{H}=(X,B)$ is a uniform hypergraph of order $n$, rank $\varrho$ and upper chromatic number $\chi_{\mathcal{H}} = \overline{\chi}$, then

$$m_{\mathcal{H}} \leq \sum_{i=2}^{\varrho} \binom{n - \overline{\chi} + 1}{i} \cdot \binom{\overline{\chi} - 1}{\varrho - i}.$$ 

Proof. It follows from the previous theorems. $\square$

5 The case of bihypergraphs

A bihypergraph is a hypergraph $H = (X,B)$ with a colouring which is a $C$-colouring and a $D$-colouring simultaneously. In other words, in such colouring every edge has three vertices coloured with precisely two colours.

Theorem 5.1 Let $\mathcal{H}=(X,B)$ be a uniform hypergraph of rank $\varrho$, $\varrho \geq 3$, and order $n$, having upper chromatic number $\chi_{\mathcal{H}} = \overline{\chi}$. If $q = Q(n, \overline{\chi})$ and $r = R(n, \overline{\chi})$, then

$$m_{\mathcal{H}} \leq \binom{n}{\varrho} - \left\lfloor \binom{q+1}{\varrho} r + \binom{q}{\varrho} (\overline{\chi} - r) \right\rfloor - \left\lfloor \binom{\overline{\chi} - 1}{\varrho} \right\rfloor + \left(\binom{\overline{\chi} - 1}{\varrho - 1}\right)(n - \overline{\chi} + 1).$$
Proof. A $\mathcal{V}$-colouring is always a $D$-colouring. Therefore the statement of Theorem 3.1 applies. Further, in any $\mathcal{V}$-colouring there are no edges having all the vertices coloured differently. Thus, by Corollary 4.1 the maximum number of edges is reached by the value of the second term of the statement, considering that the quantity

$$
\binom{\chi - 1}{\varrho} + \binom{\chi - 1}{\varrho - 1}(n - \chi + 1)
$$

is the minimum number of $\varrho$-tuples having no two vertices in the same colour class. $\square$

6 Uniform bihypergraphs of rank $\varrho = 3$

The class of bihypergraphs becomes particularly interesting when $\varrho = 3$. Indeed, in this case all edges have cardinality three, and the admissible colourings associate to them exactly two colours, with two vertices having the same colour and a third vertex having a different colour.

We recall that a Steiner triple system of order $v$ and index $\lambda$, briefly an $STS_\lambda(v)$, is a uniform hypergraph of rank three, having $v$ vertices, and such that every pair of distinct vertices is contained in exactly $\lambda$ blocks (edges). By $BSTS_\lambda(v)$ we denote an $STS_\lambda(v)$ considered as a bihypergraph. For a series of results on Steiner system as bihypergraphs see [2-10] and [13, 16].

Theorem 6.1 Let $\mathcal{H} = (X, B)$ be a uniform hypergraph of rank $\varrho = 3$, order $n$ and upper chromatic number $\chi_\mathcal{H} = \chi$. If $q = Q(n, \chi)$ and $r = R(n, \chi)$, then

$$m_\mathcal{H} \leq \binom{n}{3} - \left[ \binom{q + 1}{3} r + \binom{q}{3}(\chi - r) \right] - \left[ \binom{\chi - 1}{3} + \binom{\chi - 1}{2}(n - \chi + 1) \right].$$

Proof. It follows directly from Theorem 5.1 for $\varrho = 3$. $\square$

Theorem 6.2 Let $\mathcal{H} = (X, B)$ be a uniform hypergraph of rank $\varrho = 3$, order $n$, and upper chromatic number $\chi_\mathcal{H} = \chi$. Let $q = Q(n, \chi)$ and $r = R(n, \chi)$. 1) If $q > 2$, then

$$m_\mathcal{H} \leq \binom{n}{3} - \left[ \binom{q}{3} \frac{3r + q - 2}{q - 2} + \binom{\chi - 2}{2} \right] \cdot \frac{3n - 2\chi}{3}.$$
2) If \( q = 2 \), then
\[
m_H \leq \binom{n}{3} - \left\lceil \frac{\chi - 2}{2} \cdot \frac{3n - 2\chi}{3} \right\rceil.
\]

**Proof.** For \( q > 2 \), since:

\[
\binom{q + 1}{3} = \binom{q}{3} \cdot \frac{q + 1}{q - 2};
\]
\[
\binom{\chi - 1}{3} = \binom{\chi - 1}{2} \cdot \frac{\chi - 3}{3};
\]
the statement 1) follows from Theorem 6.1 by calculations. For \( q = 2 \), the statement 2) follows by the second of the previous formulas. □

The results of Theorem 6.2 can serve as a tool to determine the upper chromatic number of \( STSs \). Of course, these results do not seem useful in the case of systems having a small index, for example \( \lambda = 1 \), while they are important otherwise.

We conclude by providing some useful examples.

1. **STS\(_2\)(4)**

Let \( \Sigma = (X, \mathcal{B}) \) be the unique \( STS_2(4) \). It has \( n = 4 \) vertices and \( m = 4 \) blocks. It is not possible that \( \chi_{\Sigma} = 3 \). Indeed, if \( \chi_{\Sigma} = 3 \), then \( q = 1, r = 1 \) and, by Theorem 6.1
\[
m \leq \binom{4}{3} - \binom{2}{2} \cdot 2 = 2.
\]
But \( m = 4 \), a contradiction. It is easy to see that \( \chi_{\Sigma} = 2 \).

2. **STS\(_3\)(7)**

If \( \Sigma = (X, \mathcal{B}) \) is an \( STS_3(7) \), then it is not possible that \( \chi_{\Sigma} = 5 \). Indeed, in this case \( \Sigma \) has 7 vertices, \( q = 1 \) and \( r = 2 \). Then Theorem 6.1 would imply
\[
m \leq \binom{7}{3} - \binom{4}{3} - \binom{4}{2} \cdot 3 = 13,
\]
while $m = 21$; a contradiction. Therefore $\chi \leq 4.$

3. $\text{STS}_4(7)$

Let $\Sigma = (X, \mathcal{B})$ be an $\text{STS}_4(7)$, and suppose that $\chi_\Sigma \leq 4$. Since $\Sigma$ has 7 vertices, $q = 1$ and $r = 2$. From Theorem 6.1 it follows:

$$m \leq \binom{7}{3} - \binom{3}{3} - \binom{3}{2} \cdot 4 = 22,$$

while $m = 28$; a contradiction. Therefore, $\chi \leq 3$. It is known that for colourable systems $\chi_\Sigma = 3$.

References


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