A Laplace Type Problem for a Regular Lattice with
Convex-Concave Cell with Triangular Obstacles

D. Barilla

University of Messina, Department S.E.A.M.
Via dei Verdi, 75, 98122 - Messina, Italy

M. Stoka

Accademia delle Scienze di Torino, Italy

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Abstract

In the previous papers, [1], [2], [3], [4], [5], [6], [7], [8], [9], the au-
thors studies some Laplace problem for different lattices and different
obstacles. In this paper we consider two regular lattices with the cell
represented as in figure 1 and we compute the probability that a ran-
dom segment of constant length intersects a side of lattice. In particular
we obtain the probability determinated in the previous work, then the
Laplace probability.

Keywords: Geometric Probability, stochastic geometry, random sets, ran-
dom convex sets and integral geometry
1 Cells with triangle obstacles.

Let $\mathcal{R}_1(a, b, c; \alpha)$ be the regular lattice with the fundamental cell $C_0^{(1)}$ is represented as in fig. 1

\[
\text{area } C_0^{(1)} = 2ab \sin \alpha - \frac{c^2}{2} \left( \sin \alpha + \frac{1}{4} \sin 2\alpha \right). \tag{1}
\]

Considering a random segment with given length $l$ with $l < \min (a - c, b - c)$. We want compute the probability that this segment intersects a side of lattice. Evident this probability is the same of the probability $P_{\text{int}}^{(1)}$ that segment $s$ intersects the side of the fundamental cell $C_0^{(1)}$.

We denote by 0 the center of $s$, and we set $\varphi$ for the angle of $s$ and $CD$. To compute the probability $P_{\text{int}}^{(1)}$ we consider the limiting positions of $s$ for a specified value of $\varphi$ let $\tilde{C}_0^{(1)}(\varphi)$ be the determinated figure from these positions (fig. 2):
Laplace type problem

From here we can write:

\[
\text{area}_C^{(1)} (\varphi) = \text{area}_C^{(1)} - \\
[\text{area}_{a_1} (\varphi) + \text{area}_{a_2} (\varphi) + ... + \text{area}_{a_{10}} (\varphi)].
\] (2)

Considering fig. 1 and fig 2 we have that:

\[
\text{area}_{a_1} (\varphi) = \frac{cl}{2} \cos \frac{\alpha}{2} \sin \left( \varphi - \frac{\alpha}{2} \right).
\] (3)

\[
\text{area}_{a_3} (\varphi) = |C_1C_2| \cdot h_3 = \frac{cl}{2} \sin \frac{\alpha}{2} \cos \left( \varphi - \frac{\alpha}{2} \right).
\] (4)

\[
\text{area}_{a_4} (\varphi) = \left( b - \frac{c}{2} \right) \frac{l}{2} \sin \varphi.
\] (5)

\[
\text{area}_{a_5} (\varphi) = \left( b - \frac{c}{2} \right) \sin (2 \alpha - \varphi).
\] (6)

\[
\text{area}_{a_6} (\varphi) = \frac{lh_6}{2} = \frac{cl}{2} \sin \frac{\alpha}{2} \cos \left( \frac{3\alpha}{2} - \varphi \right).
\] (7)

\[
\text{area}_{a_7} (\varphi) = \left[ a - c - \frac{l}{\cos \frac{\alpha}{2}} \cos \left( \frac{3\alpha}{2} - \varphi \right) \right] \frac{l}{2} \sin (\alpha - \varphi).
\] (8)
\[ area_{a_8} (\varphi) = \frac{cl}{2} \sin \left( \frac{\pi - \alpha}{2} \right) \cos \left( \varphi - \frac{\pi}{2} + \frac{\alpha}{2} \right) = \frac{cl}{2} \cos \frac{\alpha}{2} \sin \left( \varphi + \frac{\alpha}{2} \right). \] (9)

\[ area_{a_{10}} (\varphi) = \frac{cl}{2} \sin \alpha \cos (\alpha - \varphi). \] (10)

\[ area_{a_9} (\varphi) = \frac{(b - c)l}{2} \sin (2\alpha - \varphi). \] (11)

\[ area_{a_{11}} (\varphi) = \frac{(b - c)l}{2} \sin \varphi. \] (12)

Combining (2) with (4), (5), (6), (7), (8), (9), (10), (11), (12), and (13), we obtain that

\[ area \widehat{C}_0^{(1)} (\varphi) = areaC_0^{(1)} - \left\{ \frac{l}{2} \cos \varphi [(2a - c) \sin \alpha + b \sin 2\alpha] + \right. \]

\[ \frac{l}{2} \sin \varphi \left[ 2b + \frac{c}{2} (\sin 2\alpha - \sin \alpha) + \frac{3c}{2} \cos \alpha - (b - c) \cos^2 \alpha + b \sin^2 \alpha \right] \]

\[ - \frac{l^2}{2} \sin 2(\alpha - \varphi) \right\} \] (13)

Denoting with \( M_1 \) the set of all segments \( s \) which have their center in the fundamental cell and with \( N_1 \) the set of all segments \( s \) completely contained in the fundamental cell, we have that [11]:

\[ P_{int}^{(1)} = 1 - \frac{\mu (N_1)}{\mu (M_1)}, \] (14)

where \( \mu \) is the Lebesgue measure in Euclidean plane.

The measures \( \mu (M_1) \) and \( \mu (N_1) \) are calculated using the Poincaré kinematic measure [10]

\[ dK = dx \wedge dy \wedge d\varphi, \]

where \( x, y \) are the coordinates of center point \( O \) of \( s \) and \( \varphi \) the defined angle.

To determine the limits between which the angle \( \varphi \) varies, we considered \( \varphi_1 \leq \varphi \leq \varphi_2 \), we have \( \varphi_1 = 0 \) and \( \varphi_2 + \frac{\pi}{2} - \alpha = \frac{\pi}{2} \), then \( \varphi_2 = \alpha \).

Therefore \( \varphi \in [0, \alpha] \).
We have:

$$\mu(M_1) = \int_0^\alpha d\varphi \iint_{\{ (x, y) \in C_0^{(1)} \}} dxdy = \int_0^\alpha \left( \text{area} C_0^{(1)} \right) d\varphi = \text{area} C_0^{(1)}$$ \hspace{1cm} (15)$$

and by (14),

$$\mu(N_1) = \int_0^\alpha d\varphi \iint_{\{ (x, y) \in \tilde{C}_0^{(1)}(\varphi) \}} dxdy = \int_0^\alpha \left[ \text{area} \tilde{C}_0^{(1)}(\varphi) \right] d\varphi =$$

$$\text{area} C_0^{(1)} - \int_0^\alpha \left\{ \frac{l}{2} \cos \varphi \left[ (2a - c) \sin \alpha + b \sin 2\alpha \right] +$$

$$\frac{l}{2} \sin \varphi \left[ 2b + \frac{c}{2} (\sin 2\alpha - \sin \alpha) + \frac{3c}{2} \cos \alpha - (b - c) \cos^2 \alpha$$

$$+ b \sin^2 \alpha \right] - \frac{l^2}{2} \sin 2(\alpha - \varphi) \right\} d\varphi =$$

$$\text{area} C_0^{(1)} - \left\{ \frac{l}{2} \left[ (2a - c) \sin^2 \alpha + b \sin \alpha \sin 2\alpha + 2b (1 - \cos \alpha) +$$

$$+ \frac{c}{2} (1 - \cos \alpha) (\sin 2\alpha - \sin \alpha + 3 \cos \alpha) - (b - c)(1 - \cos \alpha) \cos^2 \alpha +$$

$$+ 2b (1 - \cos \alpha) \sin^2 \alpha \right] - \frac{l^2}{4} (1 - \cos 2\alpha) \right\}. \hspace{1cm} (16)$$

The relation (1), (15), (16) and (18) give us:

$$p_{int}^{(1)} = \frac{1}{\alpha \left[ 2ab \sin \alpha - \frac{c^2}{2} (\sin \alpha + \sin 2\alpha) \right]}$$

$$\left\{ \frac{l}{2} \left[ (2a - c) \sin^2 \alpha + 2b (1 - \cos \alpha) (1 + \sin^2 \alpha) +$$

$$b \sin \alpha \sin 2\alpha + \frac{c}{2} (1 - \cos \alpha) (\sin 2\alpha - \sin \alpha + 3 \cos \alpha) \right\}. \hspace{1cm} (17)$$
\[-(b - c) (1 - \cos \alpha) \cos^2 \alpha] - \frac{l^2}{4} (1 - \cos 2\alpha) \}

For \(\alpha = \frac{\pi}{2}\) the fundamental cell becomes a rectangle with sides \(a, 2b\) and with four obstacles which are triangle isosceles rectangle same each other and the probability (19) becomes:

\[P_1 = \frac{2 \left( a + 2b - \frac{3c}{4} \right) l - l^2}{\pi \left( 2ab - \frac{c^2}{2} \right)} \]  
(18)

already found in the previous paper [1].

Evidently for \(c = 0\) the probability (24) become the Laplace probability:

\[P = \frac{2 (a + 2b) l - l^2}{2\pi ab}.\]

**References**


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