Quadratic Extended Filtering in Nonlinear Systems with Uncertain Observations

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Abstract

This paper proposes a second order polynomial estimator in nonlinear systems with uncertain observations. It is assumed that the signal and the observations are measured with an additive noise and that the observation equation includes a scalar multiplicative noise, which is described by a sequence of Bernoulli random variables that model the random interruptions which can be present in the observation mechanism. The least mean-squares error quadratic estimation problem is solved, first, by linearizing the system and, then, by defining an augmented system, in which the signal and the observation vectors are obtained by adding the corresponding second-order powers to the signal and observation vectors of the linearized system. The second order polynomial estimator in the nonlinear system is obtained from the linear filtering estimate in the augmented system.
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1 Introduction

This paper studies the problem of least mean-squared error (LMSE) quadratic estimation in nonlinear systems with uncertain observations. In this type of systems, the observation equation includes, besides the additive noise, a multiplicative noise, which models the presence or absence of the signal in the observations, described by a sequence of Bernoulli random variables.

Systems with uncertain observations are always non-Gaussian even if the noises are Gaussian and hence, the optimal estimator of the signal is not easily obtainable. This inconvenience motivates the search for suboptimal solutions under different assumptions about the signal and the noise processes involved in the model.

Signal estimation problems in non-Gaussian systems have been widely studied. De Santis et al. [11] studied the estimation problem in non-Gaussian linear systems, deducing a recursive algorithm that provides the LMSE estimator among all second-degree polynomial transformations of the observations. Since this class of transformations contains the linear ones, the estimators obtained by using this algorithm represented an improvement, in the sense of the LMSE, on those obtained by the Kalman filter in non-Gaussian systems. A generalization of this study was proposed in [1], where a recursive algorithm for an arbitrary degree polynomial estimator is proposed. Recent studies based on non-Gaussian systems can be seen in [2] and [12].

Nahi [8] was the first to address the LMSE linear estimation problem from the standpoint of uncertain observations, modelling the presence or absence of the signal in the observations using a stochastic process defined by Bernoulli variables. Considering linear systems with independent white additive noises, and under the assumption of independence of the variables that model the uncertainty, Nahi obtained a linear recursive algorithm with a structure similar to that of the Kalman filter but depending on the probability that the observations may be only noises. Later, [3] and [4] generalized the Nahi filter to the case of uncorrelated additive noises.

In nonlinear systems with uncertain observations, the volume of contributions is significantly smaller than in the case of linear systems. Some contributions to this study can be seen in the work of Nanacara and Yaz [10], Hermoso-Carazo and Linares-Pérez [5] and Nakamori et al. [9]. Recently, Wu
and Song [13] generalized the original extended Kalman filtering, the unscented Kalman filtering and the Gaussian particle filtering for nonlinear systems with uncertain observations.

The aim of this paper is to advance the study of the estimation problem in nonlinear systems with uncertain observations. To do so, we use the methodology of the extended Kalman filter in combination with the methodology used in [11], and propose a recursive algorithm that provides the LMSE quadratic filter. Specifically, we linearize the system equations and then obtain the LMSE quadratic estimator of the signal based on the linearized observations and their second-order Kronecker powers.

The paper is organized as follows: the hypotheses on the model and the linearized system are formulated in Section 2. The augmented system and the statistical properties of the augmented processes are analyzed in Section 3.1. The filtering algorithm is established in Section 3.2. Finally, in Section 4, we present a numerical simulation example to show the effectiveness of the quadratic estimator in comparison with the linearized mixture filter proposed in [5].

## 2 System model and linearized system

Let $x_k$ be the $n$-dimensional vector which describes the signal to be estimated. Let us suppose that the signal evolves in time through continuous and differentiable nonlinear functions affected by an additive noise $\{w_k; k \geq 0\}$; therefore, the equation that models its dynamics is the following:

$$x_k = f_{k-1}(x_{k-1}) + w_{k-1}, \quad k \geq 1. \quad (1)$$

In this paper, we consider $m$-dimensional nonlinear observations of the signal, $y_k$, that contain information of the same, $h_k(x_k)$, with probability $p_k$, or only consist of noise, $v_k$, with probability $1 - p_k$ (false alarm probability). Thus, by considering Bernoulli random variables $\{\gamma_k; k \geq 1\}$ with $P(\gamma_k = 1) = p_k$, the observation equation is given by

$$y_k = \gamma_k h_k(x_k) + v_k, \quad k \geq 1, \quad (2)$$

where, for all $k \geq 1$, the function $h_k$ is continuous and differentiable.

Our aim is to obtain the least mean-squared error (LMSE) quadratic estimator of the signal $x_k$ based on the observations $\{y_1, \ldots, y_k\}$. For this purpose, we consider the second-order powers, $y_j[2]$, of the observations $y_j$, $j = 1, \ldots, k$, defined by the Kronecker product [7]. Assuming that $E[y_j[2]^T y_j[2]] < \infty$, $j = 1, \ldots, k$, the quadratic filter of the signal $x_k$ is its orthogonal projection on the
space of $n$-dimensional linear transformations of $y_1, \ldots, y_k$ and their second-order powers $y_1^2, \ldots, y_k^2$.

To address the LMSE quadratic estimation problem the following hypotheses are assumed:

H.1 The initial signal, $x_0$, is a random vector with known moments up to the fourth one: $E[x_0] = \mu_0$, $Cov[x_0] = P_0$, $Cov[x_0, x_0^2] = P_0^{(3)}$ and $Cov[x_0^2, x_0^2] = P_0^{(4)}$.

H.2 The process $\{w_k; k \geq 0\}$ is a zero-mean white noise and its moments up to the fourth one are $Cov[w_k] = Q_k$, $Cov[w_k, w_k^2] = Q_k^{(3)}$ and $Cov[w_k^2, w_k^2] = Q_k^{(4)}$.

H.3 The noise $\{v_k; k \geq 1\}$ is a zero-mean white sequence with $Cov[v_k] = R_k$, $Cov[v_k, v_k^2] = R_k^{(3)}$ and $Cov[v_k^2, v_k^2] = R_k^{(4)}$.

H.4 The multiplicative noise $\{\gamma_k; k \geq 1\}$ is a sequence of independent Bernoulli variables with probabilities $P(\gamma_k = 1) = p_k$, $k \geq 1$.

H.5 The initial signal, $x_0$, and the noise processes $\{w_k; k \geq 0\}$, $\{v_k; k \geq 1\}$ and $\{\gamma_k; k \geq 1\}$ are mutually independent.

Our aim is apply the methodology of the extended Kalman filter [6], to derive a recursive algorithm to obtain a quadratic estimator of the signal $x_k$ based on the observations $\{y_1, \ldots, y_k\}$; that is, we obtain a quadratic filter, $\hat{x}_Q^{k/k}$. Accordingly, we linearize the system equations and then obtain the LMSE estimator of the signal based on these linearized observations and their second-order Kronecker powers.

At each sampling time $k \geq 1$, the linearization of the signal equation is performed by applying the Taylor expansion to $f_{k-1}$ in the filter, $\hat{x}_Q^{k-1/k-1}$, and the linearized observation equation is obtained by applying the Taylor expansion to $h_k$ in the predictor, $\hat{x}_Q^{k/k-1}$. Thus, by considering the first order terms of the corresponding expansions, the system equations are approximated by

\[
\begin{align*}
x_k &= F_{k-1}x_{k-1} + u_{k-1} + w_k, & k \geq 1, \\
y_k &= \gamma_k(H_kx_k + z_k) + v_k, & k \geq 1,
\end{align*}
\]

where $F_{k-1} = \frac{\partial f_{k-1}(x)}{\partial x} \Big|_{x=x_{k-1}^{Q/k-1}}$, $u_{k-1} = f_{k-1}(\hat{x}^{Q}_{k-1/k-1}) - F_{k-1}\hat{x}^{Q}_{k-1/k-1}$,

\[
H_k = \frac{\partial h_k(x)}{\partial x} \Big|_{x=x_{k}^{Q/k-1}} \quad \text{and} \quad z_{k} = h_k(\hat{x}^{Q}_{k/k-1}) - H_k\hat{x}^{Q}_{k/k-1}.
\]
For simplicity, we consider the centred signal and observations vectors, \( \bar{x}_k = x_k - E[x_k] \) and \( \bar{y}_k = y_k - E[y_k] \); thus, we obtain

\[
\begin{align*}
\bar{x}_k &= F_{k-1} \bar{x}_{k-1} + w_{k-1}, \quad k \geq 1, \\
\bar{y}_k &= \gamma_k H_k \bar{x}_k + \bar{v}_k, \quad k \geq 1,
\end{align*}
\]

(4)

where \( \bar{v}_k = (\gamma_k - p_k) (H_k E[x_k] + z_k) + v_k \).

Taking into account the equations of the system (3), the means \( E[x_k] \) and \( E[y_k] \), can be obtained by the following expressions:

\[
\begin{align*}
E[x_k] &= F_{k-1} E[x_{k-1}] + u_{k-1}, \quad k \geq 1, \\
E[y_k] &= p_k (H_k E[x_k] + z_k), \quad k \geq 1.
\end{align*}
\]

The statistical properties of the additive observation noise \( \{\bar{v}_k; k \geq 1\} \) are established in the following lemma:

**Lemma 2.1** The noise process \( \{\bar{v}_k; k \geq 1\} \) is a zero-mean white sequence with moments up to the fourth given by

\[
\begin{align*}
\bar{R}_k &= Cov[\bar{v}_k] = p_k (1 - p_k) c_k c_k^T + R_k, \\
\bar{R}_k^{(3)} &= Cov[\bar{v}_k, \bar{v}_k^{[2]}] = p_k (1 - p_k) (1 - 2 p_k) c_k c_k^{[2]T} + R_k^{(3)}, \\
\bar{R}_k^{(4)} &= Cov[\bar{v}_k^{[2]}, \bar{v}_k^{[2]}] = [p_k (1 - p_k) (3 p_k^2 - 3 p_k + 1) - p_k^2 (1 - p_k)^2] c_k^{[2]} c_k^{[2]T} + R_k^{(4)} + p_k (1 - p_k) (I_m^2 + K_{m^2}) [(c_k c_k^T) \otimes R_k] (I_m^2 + K_{m^2}),
\end{align*}
\]

where \( c_k = H_k E[x_k] + z_k \), \( K_{m^2} \) and \( I_{m^2} \) are the \( m^2 \times m^2 \) commutation and identity matrices, respectively, and \( \otimes \) is the Kronecker product.

Since the linear space generated by \( \{y_1, \ldots, y_k\} \) is equal to that generated by \( \{\bar{y}_1, \ldots, \bar{y}_k\} \), the quadratic estimator of \( x_k \) will be obtained from the centred observations, \( \{\bar{y}_1, \ldots, \bar{y}_k\} \), and the second-order power, \( \{\bar{y}_1^{[2]}, \ldots, \bar{y}_k^{[2]}\} \). Hence, to address the quadratic estimation problem we consider the system (4).

### 3 LMSE quadratic estimation problem

The quadratic estimation problem can be reformulated as one of linear estimation by increasing the observation vectors with their second-order Kronecker powers. For this purpose, we consider the following augmented signal and observation vectors:

\[
X_k = \begin{pmatrix} \bar{x}_k \\ \bar{x}_k^{[2]} \end{pmatrix} \in \mathbb{R}^{n + n^2}, \quad Y_k = \begin{pmatrix} \bar{y}_k \\ \bar{y}_k^{[2]} \end{pmatrix} \in \mathbb{R}^{m + m^2}.
\]
Then, the vector constituted of the first $n$ entries of the LMSE linear estimator of the augmented signal, $X_k$, based on the augmented observations, $\{Y_1, \ldots, Y_k\}$, provides the LMSE quadratic estimator of the centred original signal, $\bar{x}_k$, based on $\{\bar{y}_1, \ldots, \bar{y}_k\}$ and the required estimator will be obtained by adding the mean of the original vector, $E[x_k]$.

### 3.1 Augmented system

In this section, we analyze the dynamics of the augmented signal and observation vectors. Using the Kronecker product properties and the system hypothesis the following equations are obtained

\[
\begin{align*}
\bar{x}_k^2 &= F_k^{-1} x_{k-1}^2 + \Phi_k, \quad k \geq 1, \\
\bar{y}_k^2 &= \gamma_k H_k x_k^2 + \Psi_k, \quad k \geq 1,
\end{align*}
\]

with $\Phi_k = (I_{n^2} + K_{n^2}) [(F_k \bar{x}_k) \otimes w_k] + w_k^2$ and $\Psi_k = (I_{m^2} + K_{m^2}) [(H_k \bar{x}_k) \otimes (\gamma_k \bar{v}_k)] + \bar{v}_k^2$.

Then, the vectors $\bar{X}_k = X_k - E[X_k]$ and $\bar{Y}_k = Y_k - E[Y_k]$ satisfy the following equations:

\[
\begin{align*}
\bar{X}_k &= \bar{F}_{k-1} \bar{X}_{k-1} + \bar{W}_{k-1}, \quad k \geq 1, \\
\bar{Y}_k &= \gamma_k \bar{H}_k \bar{X}_k + \bar{V}_k, \quad k \geq 1,
\end{align*}
\] (5)

where $\bar{F}_k = \begin{pmatrix} F_k & 0 \\ 0 & F_k^2 \end{pmatrix}$, $\bar{H}_k = \begin{pmatrix} H_k & 0 \\ 0 & H_k^2 \end{pmatrix}$, $\bar{W}_k = \begin{pmatrix} w_k \\ \Phi_k - E[\Phi_k] \end{pmatrix}$, and

$\bar{V}_k = \begin{pmatrix} \bar{v}_k \\ \Psi_k - E[\Psi_k] \end{pmatrix} + (\gamma_k - p_k) H_k E[X_k]$.

In the following propositions, we study the properties of the noise processes involved in the system (5), which are required in order to tackle the linear estimation problem. Using the system hypotheses and the Kronecker product properties, the proofs can be deduced without difficulty and, therefore, are omitted here.

**Proposition 3.1** The noise $\{\bar{W}_k; k \geq 0\}$ is a zero-mean white process with

\[
\bar{Q}_k = E[\bar{W}_k \bar{W}_k^T] = \begin{pmatrix} Q_k & Q_k^{(3)} \\ Q_k^{(3)T} & Q_k^{22} \end{pmatrix},
\]

where

\[
\bar{Q}_k^{22} = (I_{n^2} + K_{n^2}) [(F_k P_k F_k^T) \otimes Q_k] (I_{n^2} + K_{n^2}) + Q_k^{(4)},
\]

and where $P_k = E[\bar{x}_k \bar{x}_k^T]$ is obtained recursively by the relation

$P_k = F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1}, \quad k \geq 1; \quad P_0 = Cov[\bar{x}_0]$. 


Proposition 3.2 The process \( \{ \bar{\mathbf{V}}_k; ~k \geq 1 \} \) has zero-mean and
\[
\mathbf{R}_{ks} = E[\bar{\mathbf{V}}_k \bar{\mathbf{V}}_s^T] = \begin{pmatrix}
\bar{R}_k \delta_{k-s} & \bar{R}_k^{12} \delta_{k-s} \\
\bar{R}_k^{12T} \delta_{k-s} & \bar{R}_k^{22} \delta_{k-s}
\end{pmatrix},
\]
where
\[
\bar{R}_k^{12} = \bar{R}_k^{(3)} + p_k (1 - p_k) c_k \text{vec}^T(P_{k,k}) H_k^{[2]T}
\]
\[
\bar{R}_k^{22} = (I_m^2 + K_{m2}) [(H_k P_{k,s} H_s^T) \otimes [\alpha_{ks} c_k c_s^T + p_k R_k \delta_{k-s}]] (I_m^2 + K_{m2}) + \bar{R}_k^{(4)} \delta_{k-s} + p_k (1 - p_k) (1 - 2p_k) [c_k^2 \text{vec}^T(P_{k,k}) H_k^{[2]T} + H_k^{[2]} \text{vec}(P_{k,k}) c_k^{[2]T}] \delta_{k-s} + p_k (1 - p_k) H_k^{[2]} \text{vec}(P_{k,k}) \text{vec}^T(P_{k,k}) H_k^{[2]} \delta_{k-s},
\]
and where \( \delta_{k-s} \) denotes the Kronecker Delta function and \( \text{vec}(\cdot) \) denotes the operator that vectorizes a matrix [7].

The matrices \( P_{k,s} = E[\bar{\mathbf{x}}_k \bar{\mathbf{x}}_s^T] \) verify \( P_{k,s} = F_{k-1} P_{k-1,s-1} F_{s-1}^T + Q_{k-1} \delta_{k-s}, \) and
\[
\alpha_{ks} = \begin{cases} 
p_k p_s (1 - p_k) (1 - p_s), & s \neq k, \\
p_k (1 - p_k)^2, & s = k.
\end{cases}
\]

Proposition 3.3
\begin{enumerate}
\item The noises \( \{ \bar{\mathbf{W}}_k; ~k \geq 0 \} \) and \( \{ \bar{\mathbf{V}}_k; ~k \geq 1 \} \) and the initial signal \( \bar{\mathbf{X}}_0 \) are uncorrelated.
\item The noise process \( \{ \gamma_k; ~k \geq 1 \} \) is independent of the process \( \{ \bar{\mathbf{W}}_k; ~k \geq 0 \}, \) with the initial condition \( \bar{\mathbf{X}}_0 \) and \( \{ \bar{\mathbf{V}}_1, \ldots, \bar{\mathbf{V}}_{k-1} \} \) for any \( k \geq 1. \)
\end{enumerate}

By using the properties established in Propositions 3.1 – 3.3, in the next Section, we obtain a recursive algorithm for the linear estimators \( \hat{\mathbf{X}}_{k/k} \) of the augmented signal \( \bar{\mathbf{X}}_k \) based on \( \{ \bar{\mathbf{Y}}_1, \ldots, \bar{\mathbf{Y}}_k \}. \)

### 3.2 Linear filtering algorithm for the augmented system

The properties of the processes involved in the system (5), studied in the previous section, do not permit us to apply the Nahi filter [8], because the additive noise of the observation equation, \( \{ \bar{\mathbf{V}}_k; ~k \geq 1 \}, \) is not white. Therefore, in this section, we derive a linear estimation algorithm for the system (5), which extends Nahi’s filter.

By the orthogonal projection lemma, the LMSE linear filter \( \hat{\mathbf{X}}_{k/k} \) of \( \bar{\mathbf{X}}_k \) satisfies the Wiener-Hopf equation
\[
E[\hat{\mathbf{X}}_{k/k} \bar{\mathbf{Y}}_i^T] = E[\bar{\mathbf{X}}_k \bar{\mathbf{Y}}_i^T], \quad \forall i = 1, \ldots, k.
\]  
(6)
To derive the algorithm we proceed according to the following stages:

**Filter in terms of the predictor:** we start from the filter expression \( \hat{X}_{k/k} \) as a linear combination of the observations

\[
\hat{X}_{k/k} = \sum_{i=1}^{k} K_{k,i} \bar{Y}_i.
\]

Then, by using the Wiener Hopf equation (6), we have

\[
E\left[ (\hat{X}_{k/k} - \bar{K}_{k,k} \bar{Y}_k) \bar{Y}_s^T \right] = E\left[ (\bar{X}_k - \bar{K}_{k,k} \bar{Y}_k) \bar{Y}_s^T \right], \quad \forall s = 1, \ldots, k - 1.
\]

Denoting the LMSE linear predictor of the signal and the observation by \( \hat{X}_{k/k-1} \) and \( \hat{Y}_{k/k-1} \), respectively, we have

\[
E\left[ (\bar{X}_k - \bar{K}_{k,k} \bar{Y}_k) \bar{Y}_s^T \right] = E\left[ (\hat{X}_{k/k-1} - \bar{K}_{k,k} \hat{Y}_{k/k-1}) \bar{Y}_s^T \right], \quad \forall s = 1, \ldots, k - 1.
\]

The following expression for the filter, \( \hat{X}_{k/k} \), in terms of the predictor, \( \hat{X}_{k/k-1} \), is obtained immediately

\[
\hat{X}_{k/k} = \hat{X}_{k/k-1} + \bar{K}_k \nu_k, \quad k \geq 1,
\]

where \( \nu_k = \bar{Y}_k - \hat{Y}_{k/k-1} \) is the innovation and \( \bar{K}_k = \bar{K}_{k,k} \) denotes the gain matrix of the filter.

Next, denoting the prediction and filtering error covariance matrices by \( P_{k/j}^X, j = k - 1, k \), from (7) and taking into account that the error \( \bar{X}_k - \hat{X}_{k/k-1} \) is uncorrelated with \( \hat{Y}_{k/k-1} \), we obtain

\[
P_{k/k}^X = P_{k/k-1}^X - \bar{K}_k \bar{\Pi}_k \bar{K}_k^T, \quad k \geq 1,
\]

where \( \bar{\Pi}_k = E\left[ \nu_k \nu_k^T \right] \) is the innovation covariance matrix.

**Predictor in terms of the filter:** considering the Wiener-Hopf equations (6) corresponding to the predictor, \( \hat{X}_{k/k-1} \), and the filter, \( \hat{X}_{k-1/k-1} \),

\[
E[\bar{X}_i \bar{Y}_i^T] = E[\hat{X}_{k-1/k-1} \bar{Y}_i^T], \quad \forall i = 1, \ldots, k - 1,
\]

\[
E[\bar{X}_{k-1} \bar{Y}_i^T] = E[\hat{X}_{k-1/k-1} \bar{Y}_i^T], \quad \forall i = 1, \ldots, k - 1,
\]

and taking into account that \( \bar{W}_k \) is centred and uncorrelated with \( \bar{Y}_i, i < k \), it is proved that

\[
\hat{X}_{k/k-1} = \bar{F}_{k-1} \hat{X}_{k-1/k-1}, \quad k \geq 1.
\]
On the other hand, from Propositions 3.1 and 3.3, the prediction error covariance matrix $P_{k/k-1}^{X}$ is recursively calculated from

$$P_{k/k-1}^{X} = \bar{F}_{k-1}P_{k-1/k-1}^{X}\bar{F}_{k-1}^{T} + \bar{Q}_{k-1}, \quad k \geq 1.$$  

**Innovation process:** to determine the innovation, $\nu_{k} = \bar{Y}_{k} - \hat{\bar{Y}}_{k/k-1}$, it is sufficient to obtain an expression for the predictor, $\hat{\bar{Y}}_{k/k-1}$, which verifies

$$E[\bar{Y}_{k}\bar{Y}_{i}^{T}] = E[\hat{\bar{Y}}_{k/k-1}\bar{Y}_{i}^{T}], \quad i = 1, \ldots, k-1.$$  

From Proposition 3.3 it follows that $\gamma_{k}$ and $(\bar{X}_{k}, \bar{Y}_{1}, \ldots, \bar{Y}_{k-1})$ are independent and hence

$$\hat{\bar{Y}}_{k/k-1} = p_{k}\bar{H}_{k}\hat{\bar{X}}_{k/k-1} + \hat{V}_{k/k-1}, \quad k \geq 1,$$

where $\hat{V}_{k/k-1}$ is the one-stage predictor of $\bar{V}_{k}$.

Therefore, the innovation, $\nu_{k}$, is given by

$$\nu_{k} = (\gamma_{k} - p_{k})\bar{H}_{k}\hat{\bar{X}}_{k} + p_{k}\bar{H}_{k}\hat{\bar{X}}_{k/k-1} + \hat{V}_{k} - \hat{V}_{k/k-1}, \quad k \geq 1. \quad (10)$$

Clearly, the innovation vectors have a zero mean and their covariance matrices, $\bar{\Pi}_{k} = E[\nu_{k}\nu_{k}^{T}]$, are given by

$$\bar{\Pi}_{k} = p_{k}(1 - p_{k})\bar{H}_{k}\bar{X}_{k} + p_{k}^{2}\bar{H}_{k}\bar{X}_{k/k-1} + \bar{V}_{k} - \bar{V}_{k/k-1}, \quad k \geq 1,$$

where $P_{k/k-1}^{V}$ is the error covariance matrix of $\hat{V}_{k/k-1}$, $P_{k,k/k-1}^{XY}$ is the error cross-covariance matrix of $\hat{X}_{k/k-1}$ and $\hat{V}_{k/k-1}$, and $P_{k,k}^{X} = E[\bar{X}_{k}\bar{X}_{k}^{T}]$, which satisfies

$$P_{k,k}^{X} = \bar{F}_{k-1}P_{k-1,k-1}^{X}\bar{F}_{k-1}^{T} + \bar{Q}_{k-1}, \quad k \geq 1; \quad P_{0,0}^{X} = P_{0/0}^{X}.$$  

**Gain matrix of the filter:** denoting the filtering and predictor errors by $\tilde{\bar{X}}_{k/k}$ and $\tilde{X}_{k/k-1}$, respectively, and taking into account the expression (7), we have

$$\tilde{\bar{X}}_{k/k} = \tilde{\bar{X}}_{k/k-1} - \bar{K}_{k}\nu_{k}, \quad k \geq 1,$$

and, since $\nu_{k}$ is uncorrelated with $\tilde{\bar{X}}_{k/k}$, clearly the gain matrix is given by

$$\bar{K}_{k} = \left[p_{k}P_{k/k-1}^{X}\bar{H}_{k}^{T} + P_{k,k/k-1}^{XY}\right]\bar{\Pi}_{k}^{-1}, \quad k \geq 1. \quad (11)$$
One-stage predictor of the observation noise: by a similar reasoning to that used to obtain expression (7), it is deduced that

$$\hat{V}_{k/l} = \hat{V}_{k/l-1} + \hat{G}_{k,l} \hat{Y}_{l/l-1}, \quad l < k, \quad k > 1,$$

(12)

where $$\hat{V}_{k/0} = E[\hat{V}_k] = 0, \quad \forall k \geq 1.$$

Furthermore, by reasoning similar to that used to obtain (11), the gain matrix $$\hat{G}_{k,l}$$ is

$$\hat{G}_{k,l} = \left[ P_l P_{k,l/l-1} \hat{A}_l^T + P_{k,l/l-1} \right] \Pi_{l-1}^{-1}.$$

The error covariance matrices of the successive prediction stages of the observation noise, $$P_{k,l/l-1}^\hat{V}$$, are obtained recursively from the following relations:

$$P_{k,l/j}^\hat{V} = P_{k,l/j-1}^\hat{V} - \hat{G}_{k,j} \Pi_j \hat{G}_{l,j}^T, \quad k, l > 1, \quad j < \min(k, l),$$

$$P_{k,l/0}^\hat{V} = R_{k,l}, \quad k, l \geq 1,$$

which are deduced from the recursive expression (12) using arguments similar to those used to obtain $$P_{k/k-1}^X$$.

Finally, we obtain the expression of the cross-covariance matrices $$P_{k,l/l-1}^{\hat{V}\hat{X}}$$ taking into account the expressions of $$\hat{V}_{k/l-1}$$ and $$\hat{X}_{l/l-1}$$, and so

$$P_{k,l/l-1}^{\hat{V}\hat{X}} = \left[ P_{k,l-1/l-2}^{\hat{V}\hat{X}} - G_{k,l-1} \Pi_{l-1} K_{l-1}^T \right] F_{l-1}^T, \quad l \leq k.$$

### 4 Numerical simulation example

This section presents a numerical simulation example to illustrate the results obtained in this paper. To do so, we considered the following nonlinear system with uncertain observations:

$$x_k = \frac{1}{x_{k-1}^2 + 3} + w_{k-1}, \quad k \geq 1$$

$$y_k = \gamma_k (x_k^2 + \exp(x_k)) + v_k, \quad k \geq 1,$$

where the following hypotheses are assumed:

i) The initial signal, $$x_0$$, is a zero-mean random variable with variance $$P_0 = 1$$, and moments up to the fourth one denoted by $$P_0^{(3)} = 0$$ and $$P_0^{(4)} = 3 - P_0^2$$.

ii) $$\{w_k; k \geq 0\}$$ is a centred white noise with variance $$Q_k = \frac{19}{3}, \quad Q_k^{(3)} = -\frac{128}{3}$$

and $$Q_k^{(4)} = \frac{123}{3} - Q_k^2.$$
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iii) \{v_k; k \geq 1\} is a centred white noise with variance \(R_k = \frac{19}{3}\), and third and fourth order moments denoted by \(R_k^{(3)} = -\frac{128}{3}\) and \(R_k^{(4)} = \frac{1123}{3} - R_k^2\), respectively.

iv) \{\gamma_k; k \geq 1\} is a sequence of independent Bernoulli random variables with \(P(\gamma_k = 1) = p\).

v) The initial signal, \(x_0\), and the noise processes \(\{w_k; k \geq 0\}, \{v_k; k \geq 1\}\) and \(\{\gamma_k; k \geq 1\}\) are mutually independent.

In this example, the performance of the extended quadratic filter algorithm is shown, together with the improvement obtained with respect to the linearized mixture filter proposed by Hermoso and Linares, [5].

Both algorithms were implemented in Matlab, being applied to the estimation of the signal in the previous nonlinear model. For each algorithm, 1000 simulations were performed, with 50 iterations for each simulation. The behaviour of the estimators was compared by the mean square errors (MSE) corresponding to the different simulations of the signal and its estimates:

\[
MSE_k = \frac{1}{1000} \sum_{s=1}^{1000} (x^s_k - \hat{x}^s_{k/k})^2, \quad k = 1, \ldots, 50,
\]

where \(x^s_k\) denotes the value of the signal generated in simulation \(s\) and iteration \(k\), and \(\hat{x}^s_{k/k}\) is the filter calculated with the corresponding algorithm in these simulations and iterations.
Figure 1: Mean-squared errors of the linearized mixture filter and the extended quadratic filter.
The performance of the extended quadratic filter was compared with that of the linearized mixture filter using different false alarm probabilities, and assuming in each case that this probability is constant in all iterations. In all the cases analyzed, the performance of the extended quadratic filter proposed in this paper was clearly superior to that of the linearized mixture of [5].

By way of example, Figure 1 illustrates the results obtained considering $P(\gamma_k = 1) = p = 0.5$. This figure shows the evolution of the MSE for both filters, reflecting, as well as the better performance of the extended quadratic filter, manifested by significantly smaller errors than those obtained by the linearized mixture, a lesser dispersion of these errors throughout the different iterations, thus confirming the effectiveness of the proposed algorithm.

We also analyzed the behaviour of the estimates obtained by the extended quadratic filter versus the false alarm probability in the observations. These results are illustrated in Figures 2 and 3.

![Figure 2: Mean-squared errors of the extended quadratic filter for $p = 0.1, 0.2, 0.3, 0.4, 0.5$.](image)
Figure 3: Mean-squared errors of the extended quadratic filter for $p = 0.5, 0.6, 0.7, 0.8, 0.9$.

As expected, these Figures show that, with an increasing value of $p$, and therefore a decreasing probability that the observations used in the estimation will lack information about the signal, the mean square error of the extended quadratic filter is significantly reduced.

Finally, on comparing these results with those shown in Figure 1 concerning the application to the linearized mixture filter, we also observe that, even for the worst case in the sense of higher mean square errors, corresponding to $p = 0.1$, the results obtained with the extended quadratic filter are better than those for the mixture filter, even for a higher false alarm probability. This again demonstrated the greater efficiency of the filter proposed in this paper, with respect to the proposal set out in [5].

5 Conclusion

The aim of this work was to advance in the study of the estimation problem in the case of noisy signals modelled by nonlinear dynamics, using the information provided by the observation of its functions, also nonlinear and also affected by noise.
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The study focused on the problem of estimation from uncertain observations. In this type of observations there exists a positive probability that they will not contain information about the signal and, therefore, will contribute only noise to the estimation.

Our study was aimed at obtaining a recursive algorithm that would provide the estimator based on second-order polynomial functions of the observations, defined by the Kronecker products. The procedure used for obtaining this estimator was to increase the original signal and observation vectors with their second-order powers, thus obtaining a new system in which we could apply a linear filtering algorithm and thus obtain the quadratic filtering algorithm in the original system.

The optimality criterion underlying this entire study was to minimize the mean-squared error. Consequently, the efficiency of the proposed algorithm is compared in terms of the mean-squared errors associated with the estimates.

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References


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