On the Signless Laplacian Energy and Signless Laplacian Estrada Index of Extremal Graphs

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Abstract

Let G be a simple (n,m)-graph. Let \( q_1, q_2, \ldots, q_n \) be the eigen values of the signless Laplacian matrix of the graph G. The signless Laplacian energy of the graph G is \( \sum_{i=1}^{n} |q_i - \frac{2m}{n}| \) and the signless Laplacian Estrada index of the graph G is \( \sum_{i=1}^{n} e^{q_i} \). In this paper we establish upperbound for Signless Laplacian energy and Signless Laplacian Estrada index of the graph G and specific the corresponding extremal graphs.

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1 Introduction

In this paper we consider all graphs are finite and simple, we say that G is a (n,m) graph if G has n vertices and m edges with vertex set \( V(G) \) and edge set \( E(G) \). The adjacency matrix of G, \( A(G) \), is a binary matrix of order n such that \( a_{ij} = 1 \) if the vertex \( v_i \) is adjacent to the vertex \( v_j \) where \( v_i, v_j \in V(G) \) and 0 otherwise. The matrix \( D(G) \) of G is the diagonal matrix of order n whose \( (i, i) \)-entry is the degree \( d_i \) of the vertex \( v_i \) in \( V(G) \). Then the matrix \( Q(G) = D(G) + A(G) \) is the Signless Laplacian matrix, for detailed spectral properties on its
see [2]. The eigenvalues $q_1, q_2, \ldots, q_n$ of the Signless Laplacian matrix $Q(G)$ of the graph $G$ are also called the signless Laplacian eigenvalues(spectrum) of $G$ and it can be ordered as $q_1(G) \geq q_2(G) \geq \ldots \geq q_n(G)$. Then the Signless Laplacian energy [7] is defined by

$$SLE = SLE(G) = \sum_{i=1}^{n} \left| q_i - \frac{2m}{n} \right|.$$ 

The Signless Laplacian Spectrum satisfy the following well-known relations [3],

$$\sum_{i=1}^{n} q_i = 2m \text{ and } \sum_{i=1}^{n} q_i^2 = 2m + \sum_{i=1}^{n} d_i^2.$$ 

The Signless Laplacian estrada index [8] is defined by

$$SLEE = SLEE(G) = \sum_{i=1}^{n} e^{q_i}.$$ 

For details on spectral graph theory see [5]. Let $k \geq 1$, we say that a graph $G$ is $k$-connected if either $G$ is the complete graph $K_{k+1}$ or $G$ has at least $k + 2$ vertices and has no $(k-1)$-vertex cut. Similarly, $G$ is $k$-edge-connected if it has at least two vertices and does not having $(k-1)$-edge cut. The connectivity of $G$ is the maximum value of $k$ for which a connected graph $G$ is $k$-connected, denoted by $\kappa(G)$. If $G$ is disconnected then $\kappa(G) = 0$. The edge-connectivity $\kappa'(G)$ is defined analogously. If $G$ is a graph of order $n$, then (i). $\kappa(G) \leq \kappa'(G) \leq n - 1$, and (ii). the three statements $\kappa(G) = n - 1$, $\kappa'(G) = n - 1$ and $G \cong K_n$ are equivalent.

Let $G_1$ and $G_2$ are any two graphs, the join, $G_1 \vee G_2$, of the graphs $G_1$ and $G_2$ is the graph obtained from the disjoint union $G_1 \cup G_2$ by adding new edges from each vertex in $G_1$ to every vertex in $G_2$. Now $G = (V(G), E(G))$ is the graph with $n$ vertices, the induced subgraph $G[V - U]$, if $U \subset V(G)$, is $G - U$.

In this paper, we determine the extremal graphs with given connectivity $k$ maximizing the Signless Laplacian Energy and Signless Laplacian Estrada index.

\section{The Signless Laplacian Energy and connectivity}

\textbf{Lemma 2.1.} [4] \newline
Let $G$ be an $(n,m)$ graph and $e$ an edge of $G$. Then, $0 \leq q_n(G) \leq q_n(G + e) \leq q_{n-1}(G) \leq q_{n-1}(G + e) \leq \ldots \leq q_1(G) \leq q_1(G + e)$.

By this lemma noting that $\sum_{i=1}^{n} q_i(G + e) - \sum_{i=1}^{n} q_i(G) = 2$. Thus we obtain the following lemma immediately.
Lemma 2.2. Let $G$ be a simple non complete graph with $n$ vertices then $SLE(G) < SLE(G + e)$ and $SLEE(G) < SLEE(G + e)$

We note that the following result for Laplacian estrada index has been given in [1].

Theorem 2.3. $G$ be a extremal graph of order $n$ with vertex connectivity $k$ and having the Signless Laplacian Energy $SLE$ (the Signless Laplacian Estrada index $SLEE$) then $G$ can be expressed as $K_k \lor (K_i \cup K_{n-k-i})$ for $1 \leq i \leq \frac{n-k}{2}$.

Proof. Let $G$ be a $(n,m)$ extremal graph with given vertex connectivity $k$. The result is clear for $k = n - 1$. Suppose that $1 \leq k \leq n - 2$, assume that $G$ has the maximal Signless Laplacian Estrada index for all connected graphs of order $n$ and vertex connectivity $k$. By hypothesis there exists a vertex cut set of order $k$, called it as $U$ such that $G - U$ is disconnected. Let the connected components of $G - U$ be $G_1, G_2, \ldots, G_r$. Suppose that $r > 2$ then adding an edge between $G_1$ and $G_2$ will preserve the connectivity of $G$ but increase the Laplacian Energy (Laplacian Estrada index) by Lemma 2.2 which gives a contradiction. Hence $r = 2$. Similarly it is true for all of $G[U]$, hence $G_1$ and $G_2$ are cliques and every vertex in $U$ is adjacent to all vertices in $G_1$ and $G_2$ in the view of Lemma 2.2. Consequently $G$ can be expressed as

$$K_k \lor (K_i \cup K_{n-k-i})$$

for $1 \leq i \leq \frac{n-k}{2}$.

This completes the proof. \hfill \Box

Recall that the first Zagreb index of a graph $G$ [6], denoted by $M_1(G)$, is defined as the sum of the squares of the degrees of the graph $G$, that is $M_1(G) = \sum_{i=1}^{n} d_i^2$.

Theorem 2.4. Let $G$ be a extremal graph of order $n$ with vertex connectivity $k$ then $SLE \leq \sqrt{n^2(n-1)(n-2) - 2(n-k-1)(n-2k-2) + nM_1(G)}$

Proof. From theorem 2.3 $G$ can be written as

$$K_k \lor (K_i \cup K_{n-k-i})$$

for $1 \leq i \leq \frac{n-k}{2}$ and clearly $G$ having $\frac{n(n-1)}{2} - i(n-k-i)$ edges, that is $2m = n(n-1) - 2i(n-k-i)$.

By Cauchy Schwarz inequality

$$\left( \sum_{i=1}^{n} x_i y_i \right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2.$$
Replacing \( x_i = \left| q_i - \frac{2m}{n} \right|, y_i = 1 \), we obtain
\[
\left( \sum_{i=1}^{n} \left| q_i - \frac{2m}{n} \right| \right)^2 \leq n \sum_{i=1}^{n} \left( q_i - \frac{2m}{n} \right)^2
= n \left[ 2m + \sum_{i=1}^{n} d_i^2 - \frac{4m^2}{n} \right]
\]
Observe that
\[
[SLE(G)]^2 \leq n^2(n-1)(n-2) + 2ni(n-k-i) - 4i^2(n-k-i)^2 + nM_1(G)
\]
Let us consider
\[f(x) = 2nx(n-k-2x) - 4x^2(n-k-x)^2,\]
then clearly
\[f'(x) = 2n(n-k-2x) + 8x^2(n-k-x) - 8(n-k-x)^2 \leq 0\]
for \(1 \leq x \leq \frac{n-k}{2}\) with equality if and only if \(x = \frac{n-k}{2}\). Thus
\[
[SLE(G)]^2 \leq n^2(n-1)(n-2) + 2n(n-k-1) - 4(n-k-1)^2 + nM_1(G).
\]
Hence
\[
SLE \leq \sqrt{n^2(n-1)(n-2) - 2(n-k-1)(n-2k-2) + nM_1(G)}
\]

3 The Signless Laplacian Estrada index and connectivity

The spectrum is the list of distinct eigen values, let it be \( \lambda_1, \lambda_2, \ldots, \lambda_t \), of any matrix of order \( n \) with their multiplicities \( m_1, m_2, \ldots, m_t \) respectively such that \( m_1 + m_2 + \ldots + m_t = n \) and it is denoted by \( (\lambda_1 \lambda_2 \ldots \lambda_t) \).

**Theorem 3.1.** Let \( G \) be a extremal graph of order \( n \) with vertex connectivity \( k \) then \( SLE \leq k e^{n-2} + (n-k-2)e^{n-3} + e^{2n+k-4} \)

**Proof.** From theorem 2.3 \( G \) can be expressed as
\[K_k \vee (K_i \cup K_{n-k-i})\]
for \(1 \leq i \leq \frac{n-k}{2}\) and clearly \( G \) having \( \frac{n(n-1)}{2} - i(n-k-i) \) edges. By an elementary computation we have, the Signless Laplacian eigen values(spectrum) of \( K_r \vee (K_s \cup K_t) \) are given as follows,
\[S[K_r \vee (K_s \cup K_t)] = \]
\[
\left( s + r + t - 2 \quad r + t - 2 \quad s + r - 2 \quad s + t - 2 + \frac{3r}{2} + \sqrt{T} \quad s + t - 2 + \frac{3r}{2} - \sqrt{T} \right)_{r \to t - 1 \quad s \to 1 \quad 1 \quad 1}
\]

where \( T = (|t - s| + \frac{t}{x})^2 + 2r \ min \{s, t\}. \)

Therefore the spectrum of the graph \( G \) is

\[
S(G) = \left( \begin{array}{cccc}
  n - 2 & n - i - 2 & i + k - 2 & n - 2 + \frac{k}{2} + \sqrt{T} \\
  k & n - k - i - 1 & i - 1 & 1 \\
  1 & 1 & 1 & 1
\end{array} \right)
\]

Now \( T = (|n - k - 2i| + \frac{k}{2})^2 + 2k \ min \{i, n - k - i\}. \) Hence

\[
SLEE = ke^{n - 2} + (n - k - i - 1)e^{n - i - 2} + (i - 1)e^{i + k - 2} + e^{n - 2 + \frac{k}{2} + \sqrt{T}} + e^{n - 2 + \frac{k}{2} - \sqrt{T}}
\]

\[
\leq ke^{n - 2} + (n - k - i - 1)e^{n - i - 2} + (i - 1)e^{i + k - 2} + e^{2n - 4 + k}
\]

Suppose

\[
g(x) = (n - k - x - 1)e^{n - x - 2} + (x - 1)e^{x + k - 2}
\]

It is clear that

\[
g'(x) = xe^{x + k - 2} - (n - k - x)e^{n - x - 2} \leq 0
\]

for \( 1 \leq x \leq \frac{n - k}{2} \). Hence we obtain the required result

\[
SLEE \leq ke^{n - 2} + (n - k - 2)e^{n - 3} + e^{2n + k - 4}
\]

\[
\square
\]

For all non zero positive values of \( a \) and \( b \), both are not lies between 0 and 1, it is clear that \( e^a + e^b \leq e^{a+b} \). Since \( e^x \) is monotonically increases in \((0, \infty)\) As well known, \( (i)\kappa(G) \leq \kappa'(G) \leq n - 1 \). Noting that \( K_k \cup (K_1 \cup K_{n-k-1}) \) has minimum degree \( k \) and edge connectivity \( k \) and the function \( ke^{n - 2} + (n - k - 2)e^{n - 3} + e^{2n + k - 4} \) is increasing with respect to \( k \). Thus the following corollaries immediately follows from Theorem 3.1.

**Corollary 3.2.** Let \( G \) be a extremal graph of order \( n \) with given edge connectivity \( k \). Then \( SLEE \leq ke^{n - 2} + (n - k - 2)e^{n - 3} + e^{2n + k - 4} \).

**Corollary 3.3.** Let \( G \) be a extremal graph with \( n \) vertices and minimum degree \( k \). Then \( SLEE \leq ke^{n - 2} + (n - k - 2)e^{n - 3} + e^{2n + k - 4} \).

**References**


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