On the Rainbow Connection for Some Corona Graphs

Dewi Estetikasari

Graduate Program of Mathematics, Faculty of Mathematics and Natural Science Universitas Andalas, Kampus Unand Limau Manis, Padang, Indonesia, 25163 estetigue@yahoo.com

Syafrizal Sy

Department of Mathematics, Faculty of Mathematics and Natural Science Universitas Andalas, Kampus Unand Limau Manis, Padang, Indonesia, 25163 syafirizalsy@gmail.com/syafrizalsy@fmipa.unand.ac.id

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Abstract

A path in an edge colored graph is said to be a rainbow path if no two edges on the path have the same color. An edge colored graph is rainbow connected if there exists a rainbow path between every pair of vertices. The rainbow connectivity of a graph G, denoted by $rc(G)$ is the smallest number of colors required to edge color the graph such that the graph is rainbow connected.

In this note, we determine the exact values of $rc(G \circ H)$ where $G$ or $H$ are complete graph $K_n$, path $P_n$, tree $T_n$, or wheel $W_n$ with $n$ is an integer.

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1 Introduction

All graphs in this paper are finite, undirected, and simple. Connectivity is perhaps the most fundamental graph-theoretic property. There are many ways to strengthen the connectivity property, such as requiring hamiltonicity, $k$-connectivity, imposing bounds on the diameter, requiring the existence of edge-disjoint spanning trees, and so on.

A natural and interesting quantifiable way to strengthen the connectivity requirement was introduced by Chartrand et al. in [1]. An edge-colored graph $G$ is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. Thus, one can properly define the rainbow connection number of a connected graph $G$, denoted $rc(G)$, as the smallest number of colors that are needed in order to make $G$ rainbow connected. Clearly that, if $G$ has $n$ vertices then $rc(G) < n$. Also notice that, clearly, $diam(G) \leq rc(G)$ where $diam(G)$ denotes the diameter of $G$.

If $P$ is a path in $G$ and $u, v \in V(P)$, then $uPv$ denotes the unique sub-path of $P$ with end vertices $u$ and $v$. If $rc(uPv) = d(u, v)$ for every $u, v \in G$, then $G$ is strongly rainbow-connected. The minimum number of color such that $G$ is strongly rainbow connected is the strong rainbow connection number, denotes $src(G)$.

2 Preliminary Notes

The corona $G \circ H$ of two graphs $G$ and $H$ (where $|V(G)| = m$ and $|V(H)| = n$) is defined as the graph $G$ obtained by taking one copy of $G$ and $m$ copies of $H$, called $H_1, \ldots, H_m$, and then joining by a line the $i$'th vertex of $G$ to every vertex in the $i$'th copy of $H$. It follows from the definition of the corona that $G \circ H$ has $m(1 + n)$ vertices. Thus, $G \circ H \not\cong H \circ G$ for any two graph $G$ and $H$.

Chartrand et al. [1] computed the precise rainbow connection number of several graph classes including complete graphs, trees, cycles, wheels.

Proposition 2.1 [1]
Let $G$ be a nontrivial connected graphs. Then
(a) $src(G) = 1$ if and only if $G$ is a complete graph,
(b) $rc(G) = 2$ if and only if $src(G) = 2$,
(c) $rc(G) = m - 1$ if and only if $G$ is a tree with $|V(G)| = m$.

Proposition 2.2 [1]
For each integer $n > 4$, $rc(Cn) = src(Cn) = \lceil \frac{n}{2} \rceil$. 
Proposition 2.3 [1]
For integer $n \geq 3$, the rainbow connection number of the wheel $W_n$ is
\[
rc(W_n) = \begin{cases} 
1, & n = 3; \\
2, & 4 \leq n \leq 6; \\
3, & n \geq 7.
\end{cases}
\]

Recently, Syafrizal Sy et al. [2] determined the exact values of Rainbow connection of fan and sun.

Theorem 2.4 [2] For integers $n, m \geq 2$, the rainbow connection number of the fan $F_n$ is
\[
rc(F_n) = \begin{cases} 
1 & \text{for } n = 2, \\
2 & \text{for } 3 \leq n \leq 6, \\
3 & \text{for } n \geq 7.
\end{cases}
\]

Corollary 2.5 [2] The rainbow connection number and strong rainbow connection number of a graph $S_n$ for $n \geq 2$ are
\[
rc(S_n) = src(S_n) = \left\lfloor \frac{n}{2} \right\rfloor + n.
\]

3 Main Results

In this section, we determine the rainbow connection numbers of corona graphs for combination of some complete graphs and tree.

Theorem 3.1 Let $G$ and $H$ be any two connected graphs,
\[
rc(G \circ H) = \begin{cases} 
1 & \text{for } (G \cong K_1 \text{ and } H \cong K_m), \\
2 & \text{for } G \cong K_1 \text{ and } H \cong P_m \text{ with } 3 \leq m \leq 6, \\
3 & \text{for } (G \cong K_1 \text{ and } H \cong P_m, m \geq 7) \text{ or } (G \cong P_2 \text{ and } H = K_m), m \geq 1.
\end{cases}
\]

Proof. We consider four cases.

Case 1. For $G \cong K_1$ and $H \cong K_m$.
Since $K_1 \circ K_m$ is a complete graph also, then by Proposition 2.1 (a), we have $rc(G \circ H) = 1$.

Case 2. For $G \cong K_1$ and $H \cong P_m$ with $3 \leq m \leq 6$.
By Theorem 2.4, we have $rc(G \circ H) = 2$.

Case 3. For $(G \cong K_1$ and $H \cong P_m, m \geq 7)$ or $(G \cong P_2$ and $H = K_m)$.
For $G \cong K_1$ and $H \cong P_m, m \geq 7$, by Theorem 2.4, we have $rc(G \circ H) = 3$.
Next, since $rc(P_2) = 1$ and $rc(K_m) = 1$ then clearly that $rc(P_2 \circ K_m) = 3$
where \( m \geq 1. \)

In the next theorem, we determine rainbow connection numbers of some corona graphs. For Case 3., a well-known class of graphs constructed from cycles are the wheels. For integer \( m \geq 3 \), suppose that \( W_m \) consists of an \( m \)-cycle \( C_m := v_1, v_2, \ldots, v_m, v_{m+1} = v_1 \) and another vertex \( v \) joined to every vertex of \( C_m \). A graph \( W_m \circ K_1 \) is defined as the graph obtained from a wheel \( W_m \) by adding \( m + 1 \) vertices and \( m + 1 \) pendant edges \( v_i v_{m+i} \) with \( v_{m+i} \) \((i = 1, \ldots, m)\) and \( vv_{2m+1} \).

**Theorem 3.2** The rainbow connection number of corona graph \( G \circ H \) is

\[
rc(G \circ H) = \begin{cases} 
  m + 1 & \text{for } (G \cong K_m \text{ and } H = K_1), \\
  2m - 1 & \text{for } G \cong T_m \text{ and } H \cong K_1, \\
  m + 3 & \text{for } G \cong W_m \text{ and } H \cong K_1, \\
  rc(G) + 3 & \text{for } |G| = m \geq 3 \text{ and } |H| = n \geq 2.
\end{cases}
\]

**Proof.** We consider four cases.

**Case 1.** For \( G \cong K_m \) and \( H = K_1 \).

By 2.1 (a), we have \( rc(K_m) = 1 \). By definition of corona graph \( K_m \circ K_1 \) and similarly to prove of rainbow connection number of sun \( rc(S_n) \), then we have \( rc(K_m \circ K_1) = m + 1 \).

**Case 2.** For \( G \cong T_m \) and \( H \cong K_1 \).

Clearly that, also \( T_m \circ K_1 \) is a tree. By definition of corona graph, the number of vertices in \( T_m \circ K_1 \) is \( 2m - 1 \). Thus, by the Proposition 2.1 (c), we have \( rc(T_m \circ K_1) = 2m - 1 \).

**Case 3.** If \( G \cong W_m \) and \( H \cong K_1 \).

To show the upper bound, first we define the coloring of \( W_m \) as Proposition 2.2 [1]. Next, let \( E = \{v_i v_{m+i} | i = 1, \ldots, m\} \cup \{vv_{2m+1}\} \) be the set of pendant edges. If all edges of \( E \) be different colored as follow \( v_i v_{m+i} \in \{rc(W_m) + 1, \ldots, rc(W_m) + m\} \), then we have \( rc(W_m \circ K_1) \leq m + 3 \).

Now, we will show that \( rc(W_m \circ K_1) \geq m + 3 \). We assume to contrary that \( rc(W_m \circ K_1) \leq m + 2 \). Let \( c \) be a strong rainbow \((m + 2)\) coloring of \( W_m \circ K_1 \). Since \( rc(W_m \circ K_1) \leq m + 2 \) and \( W_m \circ K_1 \) contains \( m \) pendant edges then there are two vertices, namely \( x \) and \( y \), such that \( xy \) is not rainbow connection in \( W_m \circ K_1 \). Thus, \( rc(W_m \circ K_1) \geq m + 2 \).

Therefore, \( rc(W_m \circ K_1) = m + 3 \).
Case 4. For \( m \geq 3 \) and \( n \geq 2 \).

We will show that \( rc(G \circ H) \geq rc(G) + 3 \). Since \( G \) is a connected graph, then \( G \) has rainbow connection number. By definition of corona graph \( G \circ H \), we have \( m \) copies of \( H \), namely \( H_1, \ldots, H_i, \ldots, H_m \). Thus, for every vertex \( u \) in \( V(H_1) \) adjacent to the vertex \( x \) in \( V(G) \). Next, since \( H_1 \) is connected graph then there is a vertex \( v \) in \( V(H_1) \) such that \( uv \in E(H_1) \). Clearly that, also \( v \) adjacent to \( x \). Since \( G \) is a rainbow connected graph then for every two vertices in \( G \), there is the rainbow path \( P \) of \( G \). Without loss of generality, similarly cases for \( y, z \in V(G) \) with \( H_i \) and \( H_m \), see Fig. 1. Consider path \( uP_v \) with \( u \in V(H_1) \) and \( v \in V(H_m) \). We consider two possible. If the color of \( c(zv) \) is 2 then \( uP_v := u, x, \ldots, y, \ldots, z, v \) such that \( uP_z := u, x, \ldots, y, \ldots, z \) is a rainbow path. This implies, \( rc(uP_v) = rc(G) + 2 \). In the other case, if the color of \( c(zv) \) is 1 then \( uP_v := u, x, \ldots, y, \ldots, z, u, v \) has \( rc(uP_v) = rc(G) + 3 \). Since for every path \( P \) of \( G \) such that \( rc(G \circ H) \) rainbow connected, this implies \( rc(G \circ H) \geq rc(G) + 3 \).

Next, to show that \( rc(G \circ H) \leq rc(G) + 3 \), without loss of generality, we may only consider a connected subgraph with three vertices in \( H_i \), \( i \in \{1, \ldots, m\} \). Let \( H_i^* \) be a connected subgraph of \( H_i \) induced by \( \{u_i, v_i, w_i\} \) for every \( i = 1, \ldots, m \). Since \( H_i^* \) is a connected subgraph then \( H_i^* \) will form a path \( P_3 \) or a cycle \( C_3 \) in \( H_i \) for every \( i = 1, \ldots, m \). By definition of corona graph, every vertex in \( H_i^* \) adjacent to a vertex \( x \in V(G) \). Thus, \( H_i^* \cup \{x\} \) consist of paths \( P_3 \) (or cycles \( C_3 \)). Now, we define a rainbow 3-coloring \( c := E(C_3) \rightarrow \{1, 2, 3\} \) of \( (\{x\} \circ H_i^*) \) such that no two edges are the colored the same. Obviously than, \( zP_x, z \in \{u_i, v_i, w_i\} \), is a rainbow path. Since \( G \) is a rainbow connected graph, then there is a vertex \( y \) such that \( zP_y \) is a rainbow path, see Fig. 2.

Next, we consider rainbow path \( zP_y \). Let \( H_j^* \) be a other connected subgraph of \( H_j \) induced by \( \{u_j, v_j, w_j\} \) for every \( j = 1, \ldots, m \). If \( c(zx) \neq c(yt) \) with
Figure 2: The illustration of $rc(G \circ H) \leq r(G) + 3$.

$t \in \{u_j, v_j, w_j\}$ then we have a rainbow path $z_x P_y t$. Conversely, if $c(zx) = c(yt)$ with $t \in \{u_j, v_j, w_j\}$, then by 3-coloring $c$ we a rainbow path $z_x P_y t$. Therefore, $rc(G \circ H) \leq rc(G) + 3$. □

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References


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