Applied Mathematical Sciences, Vol. 7, 2013, no. 98, 4875 - 4890
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2013.36346

A Way to Construct an Algorithm that Uses Hybrid Methods

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Abstract

There are wide classes of numerical methods for solving ordinary differential equations, which are involved in the requisite models for almost all of the problems of natural science. To compare these methods, scientists have proposed using some of the following criteria: stability, exactness, and stability regions. As a result, both the theoretical and practical study in the theory of numerical methods has considered implicit methods a priority. Here, it is proved that there exist explicit hybrid methods that are superior to implicit multistep methods. We have constructed general and hybrid methods to solve the initial value problem for ordinary differential equations with the order of accuracy$p \leq 10$. Furthermore, we propose an algorithm that uses these hybrid methods.

Mathematics Subject Classification: 65L

Keywords: numerical methods, hybrid methods, ordinary differential equations

1 Introduction

Numerical solutions to initial value problems in ordinary differential equations have been investigated, and such equations have many applications. Therefore, consider the following nonlinear problem:

$$y' = f(x, y), \ y(x_0) = y_0, \ x_0 \leq x \leq X. \tag{1}$$

We assume that problem (1) has a unique continuous solution on the interval $[x_0, X]$. To find the numerical solution of problem (1), the segment is divided into N equal parts with the constant step size $h > 0$, and the mesh points are defined as $x_i = x_0 + ih \ (i = 0, 1, 2, ..., N)$. The approximate and exact values of the solution to problem (1) at the mesh points $x_i \ (i = 0, 1, 2, ..., N)$ are defined in the forms y_i and $y(x_i)$, respectively. Problem (1) is classical, and therefore, investigating it has occupied the time of many famous scientists. Among them are Cauchy, Euler, Cowell, Runge, Adams and others, and some special cases of problem (1) were studied by, e.g., Newton, Leibniz, Bernoulli, and Clairaut. (See, e.g., [1, pp. 11-15].) The first direct numerical method for solving problem (1) was constructed by Euler (see [2, page 289]), and this method was developed by Adams and Runge in different directions; as a result, the classes of one-step and multistep methods appeared (see, e.g., [2, pp. 292-293]). Some authors consider the methods of Adams to be the developed form of the Runge-Kutta method. Note that whereas Adams's methods were constructed in the middle of the XXth century, the Runge-Kutta methods were developed in the late XIXth and early XXth centuries. These methods have been successfully used to address scientific and technical problems to the present day. Recently, different versions of these methods have been constructed as forward-jumping methods (see, e.g., [3], [4, pp. 146-150], [5]), hybrid methods (see [6] - [8]), semi-implicit and implicit Runge-Kutta methods, etc. (see, e.g., [9] - [14]). In the work [12], a connection between the Adams and Runge-Kutta methods was found, and in the work [15, p. 141-145], a connection between the forward-jumping methods and implicit Runge-Kutta methods was discovered. By using a wide arsenal of numerical methods to solve problem (1), we will attempt to compare some of classes of these methods, and in the second section, we will construct a hybrid method that generalizes the known multistep and hybrid methods.

2 Some comparison of the numerical methods that are applied to solve ODEs

It is known that given the increasing order of accuracy of the Runge-Kutta method, the region of its stability of the resulting approximate solutions is expanding. However, for the Adams methods, the stability region of these methods is narrowed to increase the solutions' order of accuracy. This deficiency is also found in the forward-jumping methods, but it is removed in [15, p.82-83; see also [27]] by means of a predictor and corrector scheme. Similar schemes have been previously applied to some multistep methods (see [16, p. 166-169]).

Note that until the twentieth century, scientists generally used the Euler method, as there were very few other direct methods at that time. However, at the beginning of the XXth century, scientists developed different methods to solve problem (1), and to compare these methods, they used the concept of the order of accuracy. In addition, in the middle of the XXth century, different classes of numerical methods were constructed; to compare them, scientists used the concepts of stability and regions of stability. Scientists have determined the stability of the following multistep method with constant coefficients:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} \quad (n = 0,\, 1,\, 2,\, ...,\, N-k), \tag{2}$$

which has two forms. In one of these definitions, scientists used the roots of the characteristic polynomial of method (2) (see [17]), and the other definition used the "language of ε and δ" (see [18], [19]). This method is mainly used in the definition of the convergence of some sequences, series, and methods. Although the concept of stability has been changed, the concept of limitation of the dispersion in [20] and in [21] used the concepts of stability. Method (2), which was developed by Dahlquist (see [22]), has been used by many scientists along with the work of P. Henrici (see [23]). In the paper [22], the first set defined the relationship between the order and degree for stable methods of type (2), which in [24] resulted in the famous theorem by Dahlquist. Thus, Dahlquist established the first limitation on the degree of accuracy of stable multistep methods, and this limitation can be written as $p \le 2\,[k/2] + 2$ (p is the degree of the method). To construct more accurate and stable methods, many scientists have suggested using the following method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i g_{n+i}, \tag{3}$$

here, α_i, β_i, γ_i $(i = 0, 1, 2, ..., k)$ are real numbers,

$f_m = f(x_m, y_m),\ g_m = g(x_m, y_m)$, but the function $g(x, y) = f'_x + f'_y \cdot f$,

Dahlquist proved that in the class of methods such as (3) there exist stable methods with the degree $p = 2k + 2$ (see [25]).

Each of these methods has its field of application, and the field that is appropriate for each method depends on the concept of stability and the degree of the method, which can be determined in different ways. For example, the degree of method (3) in one variable can be defined as follows:

Definition 2.1 *p is the degree of method (3) if for a sufficiently smooth function $y(x)$ the following holds:*

$$\sum_{i=0}^{k} (\alpha_i y(x + ih) - h\beta_i y'(x + ih) - h^2 \gamma_i y''(x + ih)) = O(h^{p+1}),\ h \to 0. \tag{4}$$

However, if method (3) is applied to solve the following problem:

$$y'' = F(x, y),\ y(x_0) = y_0,\ y'(x_0) = y'_0, \tag{5}$$

then asymptotic equation (4) can be rewritten as

$$\sum_{i=0}^{k} (\alpha_i y(x + ih) - h^2 \gamma_i y''(x + ih)) = O(h^{p+2}),\ h \to 0. \tag{6}$$

However, the stability of method (3) can be determined in the following form:

Definition 2.2 *Method (3) is stable if the roots of the characteristic polynomial $\rho(\lambda) \equiv \alpha_k \lambda^k + \alpha_{k-1}\lambda^{k-1} + ... + \alpha_0$ lie inside the unit circle or on its boundary and there are no multiple roots.*

If method (3) is applied to solve problem (5), it is generally true that $\beta_i = 0\ (i = 0, 1, 2, ..., k)$, and in this case, the concept of stability is defined as follows:

Definition 2.3 *Method (3) for $\beta_i = 0$ $(i = 0, 1, 2, ..., k)$ is called stable if the roots of the polynomial $\rho(\lambda)$ lie either inside the unit circle or on its boundary and there are no multiple roots except for a double root $\lambda = 1$ on its boundary (see [25]).*

Stable methods of type (3) are more accurate than stable methods of type (2). However, to apply method (3) to solve problem (1), it is necessary to calculate the function $g(x, y)$ at each step. In this case, the calculation of the functions $f'_x(x, y)$ and $f'_y(x, y)$ can be equivalent to computing the function $f(x, y)$. Thus, as a result of applying the second derivative method, the amount of computing work doubles. Hence, some authors increase the values of the degree for stable methods of type (2), and they propose using the Richardson extrapolation or a linear combination of distinct methods. One such method is to use forward-jumping methods to solve problem (1), and the general form of such methods is as follows:

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} \quad (\alpha_{k-m} \neq 0; \;\; m > 0). \tag{7}$$

The Cowell method in particular can be obtained from method (7) (see [26] or [4]). This method is generated by some of the known methods, such as the methods of Laplace and Steklov (see, e.g., [20]). In [5], it is proved that there exist stable methods of type (7) with the degree $p = k + m + 1$ $(k \geq 3m)$. The main disadvantage of methods of type (7) is the quantities $y_{n+k-m+1}$, $y_{n+k-m+2}$, y_{n+k}, which are unknown. To find the values of the quantities in [27], the predictor-corrector methods have been constructed.

Recently, solving problem (1) has involved the use of hybrid methods (see, e.g., [28]-[33]), which in one variable is written as follows:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h \sum_{i=0}^{k} \gamma_i y'_{n+i+l_i} \quad (|l_i| < 1; \; i = 0, 1, 2, ..., k) \,. \tag{8}$$

From method (8) in particular, research on hybrid methods can be seen in [28]-[33]. To construct a more accurate, stable hybrid method, consider a generalization of method (8).

3 The construction of hybrid methods with the second derivative

With a simple comparison, it can be shown that stable methods of type (3) are more accurate than stable methods of type (2). However, stable hybrid methods are also more accurate than methods of type (2) (see, e.g., [33]). We consider the construction of a hybrid method by using a method of type (3). In one variable, a hybrid method of type (3) has the following form:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h \sum_{i=0}^{k} \hat{\beta}_i y'_{n+i+\nu_i} + \; h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i} + h^2 \sum_{i=0}^{k} \hat{\gamma}_i y''_{n+i+l_i}, \tag{9}$$

where the coefficients α_i, β_i, $\hat{\beta}_i, \gamma_i$, $\hat{\gamma}_i$ $(i = 0, 1, 2, ..., k)$are real numbers and $y(x)$ is a smooth function that is determined on the interval $[x_0, X]$. Equation (9) gives some relations between the values of the functions $y(x)$ and the values of its derivatives $y'(x)$ and $y''(x)$ at the $k + 1$ mesh points. Consequently, relation (9) can be considered as a difference equation with the order of k. As a known quantity in method (9) is k, to find the degrees of methods of type (9), the relationships between the degrees p and the order k of method (9) are used. Before considering the definition of the relationship between the variables p and k for methods of type (9), we define some boundary conditions that are imposed on the coefficients of method (9), which recall the necessary condition imposed by Dahlquist on the coefficients of method (2) (see [22]). We can prove that if method (9) converges, then its coefficients satisfy the following conditions:

A. The coefficients α_i, β_i, $\hat{\beta}_i, \gamma_i$, $\hat{\gamma}_i$ $(i = 0, 1, 2, ..., k)$ are real numbers and $\alpha_k \neq 0$.

B. The sets of the roots of the polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i, \quad \vartheta(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i, \quad \hat{\vartheta}(\lambda) \equiv \sum_{i=0}^{k} \hat{\beta}_i \lambda^{i+\nu_i},$$

$$\gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^i, \quad \hat{\gamma}(\lambda) \equiv \sum_{i=0}^{k} \hat{\gamma}_i \lambda^{i+l_i}$$

are disjoint.

C. The following holds: $\rho'(1) = \vartheta(1) + \hat{\vartheta}(1) \neq 0$, $\rho''(1) \neq 0$ and $p \geq 2$. However, if $\rho'(1) = 0$, then the degree of method (9) is determined using similar asymptotic relations to (6), where $p \geq 1$ and $\rho''(1) \neq 0$.

The necessity of condition A is associated with the area action of the solutions of the considered problem. As we explore the real functions that satisfy condition A, the need for it will become obvious. Now, consider the study of condition B and assume that condition B is not satisfied. Consequently, the sets of roots for the above-mentioned polynomials intersect in at least at one point. Thus, these polynomials have a common factor, which we denote by $\varphi(\lambda)$. It is easy to understand that by using the shift operator $E^m y(x) = y(x+mh)$, the finite-difference equation can be rewritten as follows:

$$\rho(E)y_n - h\vartheta(E)y'_n - h\hat{\vartheta}(E)y'_n - h^2\gamma(E)y''_n - h^2\hat{\gamma}(E)y''_n = 0. \tag{10}$$

In view of the above-mentioned assumption, from equation (10) we have

$$\rho_1(E)y_n - h\vartheta_1(E)y'_n - h\hat{\vartheta}_1(E)y'_n - h^2\gamma_1(E)y''_n - h^2\hat{\gamma}_1(E)y''_n = 0, \tag{11}$$

where

$$\rho_1(\lambda) = \rho(\lambda)/\varphi(\lambda), \quad \vartheta_1(\lambda) = \vartheta(\lambda)/\varphi(\lambda), \quad \hat{\vartheta}_1(\lambda) = \hat{\vartheta}(\lambda)/\varphi(\lambda),$$

$$\gamma_1(\lambda) = \gamma(\lambda)/\varphi(\lambda), \quad \hat{\gamma}_1(\lambda) = \hat{\gamma}(\lambda)/\varphi(\lambda).$$

Obviously, if we denote the order of difference equation (11) by k_1, then we have $k_1 \leq k-1$. It is not difficult to understand that the difference equations (9) and (11) are equivalent, and because difference equation (11) has a unique solution, it is necessary to specify initial data. Therefore, given the k_1 initial data, finite difference equation (9) $k_1 \leq k-1$, which has the order k, must have a unique solution. However, from the theory of difference equations it is known that if the number of initial data is less than the order of the linear finite difference equations with constant coefficients, the number of solutions to these difference equations is greater than one. This fact implies that the sets of the roots of the polynomials $\rho(\lambda)$,$\sigma(\lambda)$, $\gamma(\lambda)$, $\hat{\vartheta}(\lambda)$ and $\hat{\gamma}(\lambda)$ are disjoint. Then, by considering the limit in (9) as $h \to 0$ we have:

$$\rho(1)y(x) = 0, \ \ (x = x_0 + nh). \tag{12}$$

This result implies that

$$\rho(1) = 0.$$

By taking condition (12) into account in equation (11), we obtain the following:

$$\rho_1(E)(y_{j+1} - y_j) - h\vartheta(E)y'_j - h\hat{\vartheta}(E)y'_j - h^2\gamma(E)y''_j - h^2\gamma(E)y''_j = 0, \tag{13}$$

where $\rho_1(\lambda) = \rho(\lambda)/(\lambda - 1)$.

By Lagrange's theorem, we can write:

$$y_{j+1} - y_j = hy'_j + O(h^2),$$

where we use the fact that in (13), we have

$$(\rho_1(E) - \vartheta(E) - \hat{\vartheta}(E))y'_j - h\gamma(E)y''_j - h\hat{\gamma}(E)y''_j = O(h), \ \ h \to 0. \tag{14}$$

By taking the limit as $h \to 0$, we have:

$$\rho_1(1) = \vartheta(1) + \hat{\vartheta}(1). \tag{15}$$

Thus, it is a necessary condition for the convergence of method (9) that $\rho(1) = 0$. However, if $\vartheta(\lambda) \equiv 0$ and $\hat{\vartheta}(\lambda) \equiv 0$, then $\rho(1) = \rho'(1) = 0$ is a necessary condition for the convergence of method (9).

Consider the following expansions:

$$\rho(\lambda) = \rho(1) + \rho'(1)(\lambda - 1) + \frac{1}{2}\rho''(1)(\lambda - 1)^2 + O((\lambda - 1)^3),$$

$$\vartheta(\lambda) = \vartheta(1) + \vartheta'(1)(\lambda - 1) + O((\lambda - 1)^2),$$

$$\gamma(\lambda) = \gamma(1) + \gamma'(1)(\lambda - 1) + O((\lambda - 1)^2),$$
$$y_{i+1} - y_i = hy_i' + \frac{h^2}{2} y_i'' + O(h^3).$$

These expansions are subject to conditions (15), and due to the expansions in (10), we have

$$\frac{1}{2}\rho''(1)\left(\frac{y_{j+2} - y_{j+1}}{h} - \frac{y_{j+1} - y_j}{h}\right) -$$
$$-(\vartheta'(1) + \hat{\vartheta}'(1))(y_{j+1}' - y_j') - h(\gamma(1) + \hat{\gamma}(1))y_j'' = O(h^2), \quad h \to 0. \tag{16}$$

Summarising the asymptotic equality (16) in terms of j from 0 to n, we have:

$$(\rho''(1) - 2(\vartheta'(1) + \hat{\vartheta}'(1)))(y_{n+1}' - y_0') =$$
$$= 2(\gamma(1) + \hat{\gamma}(1)) \sum_{j=0}^{n} hy_j'' - \frac{h}{2}\rho''(1)(y_{n+1}'' - y_0'') + O(h),\ h \to 0.$$

By taking the limit as $h \to 0$, we obtain:

$$(\rho''(1) - 2(\vartheta'(1) + \hat{\vartheta}'(1)))(y'(x) - y_0') = 2(\gamma(1) + \hat{\gamma}(1)) \int_{x_0}^{x} f(s, y(s))ds.$$

Furthermore, by using $y' = f(x,\, y)$ in problem (5) and setting $y''(x) = f_x'(x,\, y) + f_y'(x,\, y)f(x,\, y)$, we derive the following:

$$y'' = g(x,\, y),\ y(x_0) = y_0,\ \ y'(x_0) = y_0'. \tag{17}$$

From here, one can write

$$y'(x) = y_0' + \int_{x_0}^{x} g(s, y(s))ds.$$

By comparing these equations and utilising the fact that the solution of (17) is unique, we have:

$$\rho''(1) = 2(\vartheta'(1) + \hat{\vartheta}'(1)) + 2(\gamma(1) + \hat{\gamma}(1)).$$

It follows that if $\hat{\gamma}(1) + \gamma(1) + \hat{\vartheta}(1) + \vartheta'(1) = 0$, then$\rho''(1) = 0$. Thus, by using the substitution $z(x) = y'(x)$ in asymptotic equality (14), we have:

$$(\rho_1(E) - \vartheta(E) - \hat{\vartheta}(E))z_j - h(\hat{\gamma}(E) + \gamma(E))z_j' = O(h). \tag{18}$$

It is easy to see that due to the condition $\rho''(1) = 0$, $\lambda = 1$ is a double root of the polynomial $\rho(\lambda)$. However, asymptotic relation (18) can be regarded as approximations of the difference method

$$\sum_{i=0}^{k-1} (\bar{\alpha}_i z_{n+i} + \hat{\alpha}_i z_{n+i+\nu_i}) = h \sum_{i=0}^{k} (\gamma_i z_{n+i}' + \hat{\gamma}_i z_{n+i+\nu_i}'),$$

which for $\vartheta(1)+\hat{\vartheta}(1)=0$ or for $\rho'(1)=0$ is unstable. Therefore,

$$\vartheta(1)+\hat{\vartheta}(1)\neq 0.$$

Thus, we have proved that if method (9) converges, then $\gamma(1)+\sigma(1)\neq 0$. Now, we show that if method (11) converges, then $p\geq 2$. Indeed, if method (9) converges, then

$$\rho(1)=0,\quad \rho'(1)=\vartheta(1)+\hat{\vartheta}(1),\quad \rho''(1)=\vartheta'(1)+\hat{\vartheta}'(1)+\gamma(1)+\hat{\gamma}(1)$$

from which it follows that $p\geq 2$.

We must take into account that the basic properties of numerical methods are determined by the values of their coefficients, and thus, we consider the determination of the values of the coefficients of method (9). For this purpose, in method (9) the approximate values of the function $y(x)$ will be replaced by the exact values. Then, we have:

$$\sum_{i=0}^{k}\alpha_i y(x+ih)=h\sum_{i=0}^{k}(\beta_i y'(x+ih)+\hat{\beta}_i y'(x+(i+\nu_i)h))+$$

$$+h^2\sum_{i=0}^{k}(\gamma_i y''(x+ih)+\hat{\gamma}_i y''(x+(i+l_i)h))+R_n, \tag{19}$$

where R_n is the remainder term of method (9). If we suppose that method (9) has the degree p, then from equation (19) we have $R_n=O(h^{p+1})$.

To find the values of the quantities $\alpha_i,\ \beta_i,\ \hat{\beta}_i,\gamma_i,\ \hat{\gamma}_i,\ \nu_i,\ l_i\ (i=0,1,2,...,k)$, we use the method of undetermined coefficients, i.e., in equation (19) we use Taylor expansions of the functions $y(x)$, $y'(x)$ and$y''(x)$.

Consider the following expansions of these functions:

$$y(x+ih)=y(x)+ihy'(x)+\frac{(ih)^2}{2!}y''(x)+\ldots+\frac{(ih)^p}{p!}y^{(p)}(x)+O(h^{p+1}),$$

$$\begin{aligned}y^j(x+ih)&=y^{(j)}(x)+ihy^{(j+1)}(x)+\frac{(ih)^2}{2!}y^{(j+2)}(x)+\ldots+\\&+\frac{(ih)^{p-j}}{(p-j)!}y^{(p)}(x)+O(h^{p+1-j}),\end{aligned} \tag{20}$$

$$\begin{aligned}y^j(x+(i+m_i)h)&=y^{(j)}(x)+(i+m_i)hy^{(j+1)}(x)+\frac{((i+m_i)h)^2}{2!}\times\\&\times y^{(j+2)}(x)+\ldots+\frac{((i+m_i)h)^{p-j}}{(p-j)!}y^{(p)}(x)+O(h^{p+1-j}),\end{aligned}$$

where the quantities j and m_i $(i=0,\ 1,\ 2,\ ...,\ k)$ receive the following values:

$j=1,2$; $m_i=\nu_i$ or $m_i=l_i$ $(i=0,1,2,...,k)$.

Assume that method (9) has the degree p and consider determining the values of its coefficients. In using expansions (20) in (19), we must assume that because method (9) has the degree p, its coefficients satisfy the following conditions:

$$\sum_{i=0}^{k}\alpha_i = 0;\ \ \sum_{i=0}^{k}(\beta_i+\hat{\beta}_i) = \sum_{i=0}^{k} i\alpha_i\,,$$

$$\sum_{i=0}^{k}(\gamma_i+\hat{\gamma}_i)+\sum_{i=0}^{k}(i\beta_i+(i+\nu_i)\hat{\beta}_i) = \frac{1}{2}\sum_{i=0}^{k} i^2\alpha_i, \tag{21}$$

$$(m-1)\sum_{i=0}^{k}(i^{m-2}\gamma_i+(i+l_i)^{m-2}\hat{\gamma}_i)+\sum_{i=0}^{k}(i^{m-1}\beta_i+(i+\nu_i)\nu_i^{m-1}\hat{\beta}_i) =$$

$$= \frac{1}{m}\sum_{i=0}^{k} i^m\alpha_i \ \ (m=3,4,...,p).$$

The resulting correlation is a homogeneous system of nonlinear algebraic equations. Obviously, system (21) always has a zero (trivial) solution that is not useful for the construction of the methods addressed in this paper. Therefore, to investigate the nonzero (nontrivial) solutions of nonlinear system (21), in system (21) we observe that the number of equations is equal to $p+1$, but the number of unknowns is equal to $7k+7$. It can be assumed that systems (21) will have a nontrivial solution for the values $p \le 7k+5$. Thus, we see that to solve system (21), we can construct methods of type (9) with the degree $p \le 7k+5$. For simplicity, we consider a special case and set $k=1$. Then, using the method of (9) we obtain

$$y_{n+1} = y_n + h(\beta_0 y'_n + \beta_1 y'_{n+1}) + h(\hat{\beta}_0 y'_{n+\nu_0} + \hat{\beta}_{1^{\cdot}} y'_{n+\nu_1}) +$$

$$+h^2(\gamma_0 y''_n + \gamma_1 y''_{n+1}) + h^2(\hat{\gamma}_0 y''_{n+l_0} + \hat{\gamma}_1 y''_{n+l_1}). \tag{22}$$

In applications of method (22) to solve certain problems, it is necessary to determine the values of the quantities $y_{n+\nu_i}$ and y_{n+l_i} $(i=0,1)$. Some authors prefer to construct methods of type (9) for which the following condition holds: $\nu_i = l_i$ $(i=0,\ 1)$. In view of the conditions $\nu_i = l_i$ $(i=0,\ 1)$ from system (21), we obtain the following:

$$\beta_1+\beta_0+\hat{\beta}_1+\hat{\beta}_0 = 1,$$

$$\gamma_1+\gamma_0+\hat{\gamma}_1+\hat{\gamma}_0+\beta_1+l_1\hat{\beta}_1+l_0\hat{\beta}_0 = 1/2;$$

$$2(\gamma_1+l_1\hat{\gamma}_1+l_0\hat{\gamma}_0)+\beta_1+l_1^2\hat{\beta}_1+l_0^2\hat{\beta}_0 = 1/3;$$

$$3(\gamma_1+l_1^2\hat{\gamma}_1+l_0^2\hat{\gamma}_0)+\beta_1+l_1^3\hat{\beta}_1+l_0^3\hat{\beta}_0 = 1/4;$$

$$4(\gamma_1+l_1^3\hat{\gamma}_1+l_0^3\hat{\gamma}_0)+\beta_1+l_1^4\hat{\beta}_1+l_0^4\hat{\beta}_0 = 1/5;$$

$$
\begin{aligned}
5(\gamma_1 + l_1^4\hat{\gamma}_1 + l_0^4\hat{\gamma}_0) + \beta_1 + l_1^5\hat{\beta}_1 + l_0^5\hat{\beta}_0 &= 1/6;\\
6(\gamma_1 + l_1^5\hat{\gamma}_1 + l_0^5\hat{\gamma}_0) + \beta_1 + l_1^6\hat{\beta}_1 + l_0^6\hat{\beta}_0 &= 1/7;\\
7(\gamma_1 + l_1^6\hat{\gamma}_1 + l_0^6\hat{\gamma}_0) + \beta_1 + l_1^7\hat{\beta}_1 + l_0^7\hat{\beta}_0 &= 1/8;\\
8(\gamma_1 + l_1^7\hat{\gamma}_1 + l_0^7\hat{\gamma}_0) + \beta_1 + l_1^8\hat{\beta}_1 + l_0^8\hat{\beta}_0 &= 1/9;\\
9(\gamma_1 + l_1^8\hat{\gamma}_1 + l_0^8\hat{\gamma}_0) + \beta_1 + l_1^9\hat{\beta}_1 + l_0^9\hat{\beta}_0 &= 1/10.
\end{aligned}
\tag{23}
$$

By solving the resulting system, one can construct different methods with the degree $p \leq 10$.

$$
\begin{array}{ll}
\beta_1 = 0, & \beta_0 = 0.22067064832761202,\\
\hat{\beta}_0 = 0.6162572072176795, & \hat{\beta}_1 = 0.1630721444547081,\\
\gamma_1 = 0, & \gamma_0 = 4.674947171097215,\\
\hat{\gamma}_1 = 0.05722827972230951, & \hat{\gamma}_0 = -4.656220896767483,\\
\nu_1 = 0.6088582789777904, & \nu_0 = -0.00013437381414762,\\
l_1 = 0.9359620492093735, & l_0 = 0.44042666646102624;
\end{array}
\tag{24}
$$

$$
\begin{array}{ll}
\beta_1 = 0, & \beta_0 = 0.09912521838115614,\\
\hat{\beta}_0 = 0.5357638296766549, & \hat{\beta}_1 = 0.36511095194218884,\\
\gamma_1 = 0.00280716584367082, & \gamma_0 = 4.674947171097215,\\
\hat{\gamma}_1 = -0.03262601624915164, & \hat{\gamma}_0 = 0.05344430292591017,\\
\nu_1 = 0.8396664715066857, & \nu_0 = 0.39573654170453365,\\
l_1 = 0.9199230019069407, & l_0 = 0.2622435049872752;
\end{array}
\tag{25}
$$

$$
\begin{array}{ll}
\beta_1 = 0.135344792587737, & \beta_0 = 0.17243539501678937,\\
\gamma_1 = 0.00524046532392486, & \gamma_0 = 0.00896517524992092,\\
\hat{\gamma}_1 = -0.03262601624915164, & \hat{\gamma}_0 = 0.0755529944561066,\\
\nu_1 = 0.6449489742783179, & \nu_0 = 0.6010234186344573,\\
l_0 = 0.4122642806807163. &
\end{array}
\tag{26}
$$

Consider the application of the method encapsulated in (24)-(26) to solve the following problems:

1. $y' = \cos x$, $y(0) = 0$, $0 \leq x \leq 1$, where the user decides to move $h = 0,1$. The results can be found in Table 1.

Number of example	x	*Method (24)*	*Method (25)*	*Method (26)*
I	010	0.73E-12	0.28E-05	0.1E-04
	0.40	0.27E-11	0.11E-04	0.1E-03
	0.70	0.42E-11	0.18E-04	0.28E-03
	1.00	0.52E-11	0.23E-04	0.52E-03

These results indicate that the method of (24) is more accurate than other methods; however, note that using only one example cannot prove this point.

According to this result, method (24) - (26) can be applied to obtain the following solution to the problem which solved in [28] by hybrid method with degree$p = 7$:

2. $y' = -y,\ y(0) = 1,\ 0 \le x \le 1$, (the exact solution is $y(x) = \exp(-x)$).

3.$y' = 8(x - y) + 1,\ y(0) = 2,\ 0 \le x \le 1$ (the exact solution is $y(x) = x + 2\exp(-8x)$). The result is presented in Tables 2 and 3.

Number of example	x	*Method (24)*	*Method (25)*	*Method (26)*
I	010	0.4E-12	0.26E-05	0.93E-04
	0.40	0.12E-11	0.79E-05	0.27E-03
	0.70	0.15E-11	0.1E-04	0.36E-03
	1.00	0.16E-11	0.1E-04	0.38E-03

Number of example	x	*Method (24)*	*Method (25)*	*Method (26)*
I	010	0.15E-06	0.19E-02	0.6E-02
	0.40	0.57E-07	0.71E-03	0.22E-02
	0.70	0.46E-08	0.11E-03	0.56E-03
	1.00	0.11E-08	0.14E-04	0.46E-04

Remark. Many scholars argue that those methods that are constructed by taking into account some properties of the considered problem usually give the best results. Indeed, these properties can strongly change the methods that are used to solve them. Thus, we consider the solution of (17). This problem is defined in general terms as follows:

$$y''(x) = f'_x(x, y) + f'_y(x, y) \cdot y'.$$

Hence, problem (17) has the following form:

$$y'' = F(x, y, y'),\ y(x_0) = y_0,\ y'(x_0) = y'_0. \tag{27}$$

If in (27) we take into account the differential equation from problem (1), then the function $F(x, y, y')$ is independent of y', and as a result, we obtain that $F(x, y, y') = g(x, y)$. It is obvious that the solution of (27) is necessary for the construction of methods that find approximate values of the function $y(x)$ and its first derivative at the mesh points x_i $(i = 0, 1, ..., N)$. Thus, we see that to solve problem (27), it is necessary to use systems that consist of two methods. If implicit methods are incorporated, then the number of quantities derived from the two chosen methods increases. However, if we want to solve problem (17), the necessity of computing the values of the function $y'(x)$ is not apparent. Therefore, by using only one method, we can solve problem (17). Usually in such a case, the Stoermer method or its modifications are appropriate, and one can easily see that in a particular case from method (9), the following can be obtained:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \beta_i g_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i g_{n+i+l_i}, \tag{28}$$

which is a generalization of the Stoermer method. Naturally, method (28) is more accurate than the method of Stoermer. However, in using method (28), there are difficulties in determining the values of y_{n+i+l_i} $(i = 0, 1, 2, ..., k)$; these difficulties can be resolved with the help of the block method (see, e.g., [34]). Thus, it can be shown that solving problem (1) can be replaced with solving equivalent problem (17). If we choose to solve problem (17) with method (9), then we achieve the best results. Indeed, the replacement of solving differential equations with solving other differential equations that are of higher orders is often useful.

4 Conclusion

This study examined a type of hybrid multistep method that is more accurate than the known methods. However, the application of such methods is accompanied by some difficulties. To overcome these difficulties, one can use the more exact numerical methods. However, although constructing specific algorithms for the methods of type (9) does not cause extra problems, selecting corresponding methods with the required accuracy can cause problems. We remark that in decreasing the values for the degrees of methods of type (9), these difficulties also increase. By using the fact that methods of type (9) have the degree $p \leq 7k + 5$, some of the coefficients in method (9) can be chosen such that the method has some of the properties of the solution for the initial problems. Note that we have constructed three different methods that have the same degree, which are applied to solve a particular problem. The result derived with method (24) is superior to the result of other received methods. To find the maximal degree of method (9), one can use the criteria defined in [5]. Among the above-mentioned methods, only the coefficients of method (24) satisfy the criteria from [5].

ACKNOWLEDGEMENTS. The authors with express their thanks to academician Ali Abbasov for his suggestion that the investigate the computational aspects. This work was supported by the Science Development Foundation of Azerbaijan (Grand EIF-2011-1(3)-82/27/1).

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Received: June 7, 2013