A Generalized Retarded Gronwall-like Inequalities

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Abstract

In this paper, we establish a generalized retarded integral inequality of Gronwall-like type. The results we obtained can be used as handy tools in discussing the behavior of differential equations and integral equations.

Keywords: inequality, differential equation, integral equation

1 Introduction

Recently, many authors have made researches on retarded integral inequalities and obtained plenteous results. In 2006, Olivia Lipovan gave the following inequalities:

Lemma 1.1. ([5],Theorem 1.1) Let $k \in C(R_+ , R_+ )$, $\alpha \in C^1 (R_+ , R_+ )$, $a \in C(R_+ \times R_+ , R_+ )$ with $(t,s) \longmapsto \partial_t a(t,s) \in C(R_+ \times R_+ , R_+ )$. Assume in addition that $\alpha$ is nondecreasing with $\alpha(t) \leq t$ for $t \geq 0$. If $u \in C(R_+ , R_+ )$ satisfies

$$
u(t) \leq k(t) + \int_0^{\alpha(t)} a(t,s)u(s)ds, t \geq 0
$$

Then

$$
u(t) \leq k(t) + e^{\int_0^{\alpha(t)} a(t,s)ds} \int_0^t e^{-\int_0^{\alpha(r)} a(r,s)ds} \partial_t (\int_0^{\alpha(r)} a(r,s)k(s)ds)dr, t \geq 0.
$$

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Motivated by the works of Pachpatte [1] and Olivia Lipovan [5], the purpose of this paper is to discuss some more general integral inequalities.

## 2 Main Results

Throughout $R_0^+ = [0, \infty]$, $R^+ = (0, \infty)$, and we use the notion $\text{Dom}(f)$ to denote the domain of a function $f$.

**Theorem 2.1.** Let $a \in (R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t) \leq t$ on $R_0^+$, $f, g, h \in C(R_0^+, R_0^+)$. If $u \in C(R_0^+, R_0^+)$ satisfies

$$u(t) \leq a(t) + \int_0^t h(s)u(s)ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)u(\tau)d\tau ds, t \in R_0^+, \quad (1)$$

then we have

$$u(t) \leq a(t)exp(\int_0^t h(s)ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)d\tau ds), t \in R_0^+. \quad (2)$$

**Proof.** Since $a(t)$ is positive and nondecreasing, from (1) we have

$$\frac{u(t)}{a(t)} \leq 1 + \int_0^t h(s)\frac{u(s)}{a(t)}ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)\frac{u(\tau)}{a(t)}d\tau ds$$

$$\leq 1 + \int_0^t h(s)\frac{u(s)}{a(s)}ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)\frac{u(\tau)}{a(\tau)}d\tau ds$$

We define a function $z(t)$ on $R_0^+$ by

$$z(t) = 1 + \int_0^t h(s)\frac{u(s)}{a(s)}ds + \int_0^{\alpha(t)} f(s) \int_0^s g(\tau)\frac{u(\tau)}{a(\tau)}d\tau ds$$

Then $z(t)$ is positive and nondecreasing, $z(0) = 1$, $\frac{u(t)}{a(t)} \leq z(t), t \in R_0^+$ and

$$z'(t) = h(t)\frac{u(t)}{a(t)} + f(\alpha(t))\alpha'(t)\int_0^{\alpha(t)} g(s)\frac{u(s)}{a(s)}ds$$

$$\leq h(t)z(t) + f(\alpha(t))\alpha'(t)\int_0^{\alpha(t)} g(s)z(s)ds$$

$$\leq h(t)z(t) + f(\alpha(t))\alpha'(t)z(t)\int_0^{\alpha(t)} g(s)ds$$

i.e.

$$\frac{z'(t)}{z(t)} \leq h(t) + f(\alpha(t))\alpha'(t)\int_0^{\alpha(t)} g(s)ds$$
Integrating the above relation from 0 to $t$, we have
\[ z(t) \leq z(0) \exp \left( \int_0^t h(s) \, ds + \int_0^{\alpha(t)} f(s) \, g(\tau) \, d\tau \, ds \right) \]
\[ = \exp \left( \int_0^t h(s) \, ds + \int_0^{\alpha(t)} f(s) \, g(\tau) \, d\tau \, ds \right) \]

Since $\frac{u(t)}{u(t)} \leq z(t)$, we have
\[ u(t) \leq a(t) \exp \left( \int_0^t h(s) \, ds + \int_0^{\alpha(t)} f(s) \, g(\tau) \, d\tau \, ds \right) \]

So the relation (2) is true.

**Remark 1.** Setting $\alpha(t) \equiv 0$ in Theorem 2.1, we obtain the well-known Gronwall-Bellman inequality [3,4].

**Theorem 2.2.** Let $f(t,s), g(t,s), h(t,s)$ be continuous on $(R_0^+ \times R_0^+, R_0^+)$ and nondecreasing in $t$ for every $s$ fixed. Moreover, let $a \in (R_0^+, R^+)$ and $\alpha \in C^1(R_0^+, R_0^+)$ be nondecreasing with $\alpha(t) \leq t$. If $u \in C(R_0^+, R_0^+)$ satisfies

\[ u(t) \leq a(t) + \int_0^t h(t,s)u(s) \, ds + \int_0^{\alpha(t)} f(t,s)(u(s) + \int_0^s g(s,\tau)u(\tau) \, d\tau) \, ds, \quad t \in R_0^+, \tag{3} \]

then

\[ u(t) \leq a(t) \exp \left( \int_0^t h(t,s) \, ds + \int_0^{\alpha(t)} f(t,s)(1 + \int_0^s g(s,\tau) \, d\tau) \, ds \right), \quad t \in R_0^+. \tag{4} \]

**Proof.** Letting $t = 0$ in (3), the result inequality (4) holds trivially. Fixing an arbitrary number $t_0 \in R_0^+$, we define a positive and nondecreasing function $z(t)$ by

\[ z(t) = a(t_0) + \int_0^t h(t_0,s)u(s) \, ds + \int_0^{\alpha(t)} f(t_0,s)(u(s) + \int_0^s g(s,\tau)u(\tau) \, d\tau) \, ds. \]

Then $z(0) = a(t_0)$, $u(t) \leq z(t)$, $t \in [0,t_0]$. Since $\alpha(t) \leq t$, we have
\[
z'(t) = h(t_0,t)u(t) + f(t_0,\alpha(t))\alpha'(t)(u(\alpha(t)) + \int_0^{\alpha(t)} g(\alpha(t),s)u(s) \, ds)
\leq h(t_0,t)z(t) + f(t_0,\alpha(t))\alpha'(t)(z(\alpha(t)) + \int_0^{\alpha(t)} g(\alpha(t),s)z(s) \, ds)
\leq h(t_0,t)z(t) + f(t_0,\alpha(t))\alpha'(t)(z(t) + \int_0^{\alpha(t)} g(\alpha(t),s)z(s) \, ds) \]
\[
\leq h(t_0,t)z(t) + f(t_0,\alpha(t))\alpha'(t)(z(t) + \int_0^{\alpha(t)} g(\alpha(t),s)z(s) \, ds) \]

...
i.e.
\[
\frac{z'(t)}{z(t)} \leq h(t_0, t) + f(t_0, \alpha(t))\alpha'(t)(1 + \int_0^{\alpha(t)} g(\alpha(t), s)\,ds).
\]
Integrating the above relation on \([0, t_0]\) yields
\[
z(t_0) \leq z(0)\exp\left(\int_0^{t_0} h(t_0, s)\,ds + \int_0^{t_0} f(t_0, \alpha(s))\alpha'(s)(1 + \int_0^{\alpha(s)} g(\alpha(s), \tau)d\tau)\,d\tau\right).
\]
Since \(t_0\) is arbitrary, using \(u(t) \leq z(t)\) on \([0, t_0]\), then taking \(t = t_0\) on \(\mathbb{R}^+_0\), we get
\[
u(t) \leq a(t)\exp\left(\int_0^t h(t, s)\,ds + \int_0^t f(t, \alpha(s))\alpha'(s)(1 + \int_0^{\alpha(s)} g(\alpha(s), \tau)d\tau)\,d\tau\right)
= a(t)\exp\left(\int_0^t h(t, s)\,ds + \int_0^{\alpha(t)} f(t, s)(1 + \int_0^s g(s, \tau)d\tau)\,ds\right).
\]
So the result is true. \(\square\)

**Remark 2.** Let \(h(t, s) \equiv 0\) and \(g(t, s) \equiv 0\) in Theorem 2.2, If
\[
u(t) \leq a(t) + \int_0^{\alpha(t)} f(t, s)u(s)\,ds,
\]
holds, then the result is \(u(t) \leq a(t)\exp\int_0^{\alpha(t)} f(t, s)\,ds, t \in \mathbb{R}^+_0\).

Compared with the inequality discussed in Lemma 1.2, we obtain a conciser conclusion by different method. Moreover, if \(a(t) \equiv t\) in Remark 2, we obtain the Corollary 1.1 in paper [5].

**Remark 3.** Let \(f, a, \alpha\) be as in Theorem 2.2, \(h(t, s) \equiv 0\) and \(g(t, s) \equiv 0\). Suppose \(u \in \mathcal{C}(\mathbb{R}^+_0, \mathbb{R}^+_0)\) is a solution to integral equation
\[
u(t) = a(t) + \int_0^t h(t, s)u(s)\,ds + \int_0^{\alpha(t)} f(t, s)(u(s) + \int_0^s g(s, \tau)u(\tau)d\tau)\,d\tau, t \in \mathbb{R}^+_0.
\]
If
\[
\lim_{t \to \infty} \sup a(t) < \infty,
\]
and
\[
\int_0^t h(t, s)\,ds, \int_0^t f(t, s)\,ds, \int_0^t g(t, s)\,ds < \infty,
\]
then \(u\) is bounded.

Our result generalizes and improves the Corollary 1.2 in paper [5].
Theorem 2.3. Let $a \in (R^+_0, R^+)$ and $\alpha_i \in C^1(R^+_0, R^+_0)$ be nondecreasing with $\alpha_i(t) \leq t$ on $R^+_0$, $f_i, g_i, h \in C(R^+_0, R^+_0)$. If $u \in C(R^+_0, R^+_0)$ satisfies

$$u(t) \leq a(t) + \int_0^t h(s)u(s)ds + \sum_{i=1}^n \int_0^{\alpha_i(t)} f_i(s) \int_0^{\beta_i(s)} g_i(\tau)u(\tau)d\tau ds, t \in R^+_0, \quad (5)$$

then

$$u(t) \leq a(t) \exp \left(\int_0^t h(s)ds + \sum_{i=1}^n \int_0^{\alpha_i(t)} f_i(s) \int_0^{\beta_i(s)} g_i(\tau)d\tau ds\right), t \in R^+_0. \quad (6)$$

Theorem 2.4. Let $f_i(t, s), g_i(t, s), h(t, s)$ be continuous on $R^+_0 \times R^+_0$ and nondecreasing in $t$ for every $s$ fixed. Moreover, let $a \in (R^+_0, R^+)$ and $\alpha_i \in C^1(R^+_0, R^+_0)$ be nondecreasing with $\alpha_i(t) \leq t$. If $u \in C(R^+_0, R^+_0)$ satisfies

$$u(t) \leq a(t) + \int_0^t h(t, s)u(s)ds + \sum_{i=1}^n \int_0^{\alpha_i(t)} f_i(t, s)(u(s) + \int_0^{s} g_i(s, \tau)u(\tau)d\tau)ds, t \in R^+_0, \quad (7)$$

then

$$u(t) \leq a(t) \exp \left(\int_0^t h(t, s)ds + \sum_{i=1}^n \int_0^{\alpha_i(t)} f_i(t, s)(1 + \int_0^{s} g_i(s, \tau)d\tau)ds\right), t \in R^+_0. \quad (8)$$

Since the proof of Theorem 2.3 and Theorem 2.4 follows by the similar argument as in the proof of Theorem 2.1 and Theorem 2.2. We omit the details.

3 Some Applications

Corollary 1. Let function $f, g, h, a, \alpha$ be as in Theorem 2.1. Suppose $u \in C(R^+_0, R^+_0)$ is a solution to Volterra integral equation

$$u(t) = a(t) + \int_0^t h(s)u(s)ds + \int_0^{\alpha(t)} f(s) \int_0^{s} g(\tau)u(\tau)d\tau ds, t \in R^+_0.$$

If

$$\lim_{t \to \infty} \sup a(t) < \infty,$$

and

$$\int_0^{\alpha(\infty)} f(s)ds, \int_0^{\infty} g(s)ds, \int_0^{\infty} h(s)ds < \infty$$

Then $u$ is bounded.
References


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