A Note on Visual Pattern Recognition
by Gaussian Curvature Mapping

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Abstract
Biometry is any technique that allows the measurement of vital characteristics or physical properties of a person. More specifically deals with understand how some of these characteristics of the human body, unique to each individual, may be used as a tool for individual recognition. The application of biometry in the study of physical characteristics, allows to identify a number of factors which enable the identification of a subject, providing a response in probabilistic terms. It is possible to recognize a person by a map of Gaussian curvature of his face? In this letter we want to show a possible approach.

Keywords: Biometry, Gaussian Curvature

1 Introduction
The processes of pattern recognition are crucial for technological systems and also for neuroscience [4],[12]. The objective of biometric systems is the recognition of individuals based on some physical characteristics that are unique for
Elmo Benedetto and Ignazio Licata

each individual. The bidimensional facial recognition uses information from a two-dimensional image of a face and relies on the comparison of relative positions of facial features. There have been many studies regarding the two-dimensional biometrics but, more recently, advances in techniques for three-dimensional facial recognition are very promising and interesting. It should also be noted that facial recognition is the only viable recognition technology able to operate without the subject’s cooperation, since facial characteristics can be captured from video cameras or closed-circuit television. Let us remember that visual recognition is not only a technology research, but it is also one of the essential biological activity. What we want to suggest in this note is that the development of 3D technology according to a PDE approach follows a path of biological plausibility. Indeed, it is shown that the neural activity of recognition that realized in the below temporal cortex is based on the identification and progressive selection of a minimum number of invariants. They can be characterized as constraints on PDE systems. In this paper we show, following Elyan and Ugail, that the Gaussian mapping between these constraints is that requires a minimum number of input and strongly characterizes the topology of a face, suggesting a new line of crossing between natural and artificial recognition.

2 PDE Approach

The use of partial differential equations (PDE) has long been considered a powerful tool for geometric modeling. For example Bloor and Wilson introduced a method that defines smooth surfaces as solutions of elliptic PDEs [3]. The authors propose a model based on PDE and aimed at the reconstruction of the human face geometry. The problem of the reproduction of a human face geometry, which can really be considered realistic and functional, is undoubtedly of great relevance. The PDE method has the advantage that most of the information defining an object comes from its boundaries. This permits an object to be generated and controlled by a very few parameters such as boundary-value conditions and global coefficients associated with an elliptic PDE [5]. We start following the approach of Elyan and Ugail [6] where ”the geometry corresponding to a human face is treated as a set of surface patches, whereby each surface patch is represented using four boundary curves in the 3-space that formulate the appropriate boundary conditions for the chosen PDE. These boundary curves are extracted automatically using 3D data of human faces obtained using a 3D scanner. The solution of the PDE generates a continuous single surface patch describing the geometry of the original scanned data.” By using a parametric representation of surface $X = X(u, v)$, defined on a finite domain $S \subset \mathbb{R}^2$, and by specifying boundary data around the edge region of $\partial S$, the surface is regarded as a solution of a PDE of the form
By assuming a suitable periodicity for the analytical solution, we can restrict the domain of solution $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$. The boundary conditions imposed on the solution are of the form $X(0, v) = P_0(v)$, $X(s, v) = P_s(v)$, $X(t, v) = P_t(v)$, $X(1, v) = P_1(v)$ where $P_0(v)$ and $P_1(v)$ are the edges of the surface at $u = 0$ and $u = 1$. With this hypothesis, the analytical solution of the equation (1) is

$$X(u, v) = \tilde{X}(u) \cos(nv) + \tilde{X}(u) \sin(nv) + R(u, v)$$

where $n$ is an integer and $\tilde{X}(u) = c_1e^{nu} + c_2ue^{nu} + c_3e^{-nu} + c_4ue^{-nu}$ with

$$c_1 = \left[-P_0(2n^2e^{2n} + 2ne^{2n} + e^{2n} - 1) + P_1(ne^{3n} + e^{3n} + ne^n - e^n) - 2P_sne^{2n} - P_t(e^{3n} - e^n)\right]/d$$

$$c_2 = \left[P_0(2n^2e^{2n} + ne^{2n} - n) + P_1(ne^{3n} + 2n^2e^n - ne^n) - P_s(2ne^{2n} - e^{2n} + 1) + P_t(e^{3n} - 2ne^n - e^n)\right]/d$$

$$c_3 = \left[P_0(e^{4n} - 2n^2e^{2n} + 2ne^{2n} - e^{2n}) - P_1(ne^{3n} + e^{3n} + ne^n - e^n) + 2P_sne^{2n} + P_t(e^{3n} - e^n)\right]/d$$

$$c_4 = \left[P_0(ne^{4n} + 2n^2e^{2n} - ne^{2n}) - P_1(2n^2e^{3n} + ne^{3n} - ne^n) + P_s(e^{4n} - 2ne^{2n} - e^{2n}) + P_t(2ne^{3n} - e^{3n} + e^n)\right]/d$$

with $d = e^{4n} - 4n^2e^{2n} - 2e^{2n} + 1$.

$R(u, v)$ is computed using the spectral approximation methods based on the difference between a finite Fourier series representing a given boundary condition and the original data corresponding to that boundary condition. The above methodology is used to create complex geometric shapes using the resolution of a PDE coupled with an appropriate set of boundary conditions. In particular, the facial geometry object of reconstruction is understood as a set of surface patches, each resulting from the integration of a PDE and the use of four appropriate boundary curves obtained through three-dimensional facial scans.

3 Gaussian Curvature of Face

Let us remember that if we have a surface in the 3-dimensional Euclidean space, we can choose, at each point $P$, a unit normal vector. A normal plane at $P$, is a plane that contains the normal and cut the surface in a plane.
curve. Of all the normal sections passing through a given point, there are two that are perpendicular and the corresponding curves have the maximum and minimum values of curvature, named principal curvatures. The product of the principal curvatures is the gaussian curvature of the surface at the point. The curvature of a curve is by definition the reciprocal of the radius of the osculating circle. We take three points $P'$, $P$ and $P''$ on the curve. For these points passes only a circumference and we do bring $P'$ and $P''$ to $P$. When the points are infinitely close, we get the osculating circle and the inverse of its radius is for definition the curvature of the curve at that point $P$.

It is well known that most familiar vector algebras involve an inner product between vectors. This is a rule which associates a number, the dot product, with two vectors. It is a linear function of both vectors, called metric tensor, and it is used to define the length of, and angle between, vectors. A surface equipped with a metric tensor is known as a Riemannian surface. By integration, the metric tensor allows one to define and compute the length of curves on the surface. On a Riemannian surface, there is a canonical connection called the Levi-Civita connection that provides a well-defined method for differentiating vector fields or any other kind of tensor. The connection coefficients in a coordinate basis are also called Christoffel symbols. The Christoffel symbols may be used for performing practical calculations in differential geometry. For example, the Riemann curvature tensor can be expressed entirely in terms of the Christoffel symbols and their first partial derivatives. Therefore, from the analytic solution of equation we can deduce its metrics, its Levi-Civita connections and finally its curvature. Now we consider the solution (2) and the following derivative

$$\frac{\partial X(u,v)}{\partial u} = \frac{\partial \tilde{X}(u)}{\partial u} \cos(nv) + \frac{\partial \tilde{X}(u)}{\partial u} \sin(nv) + \frac{\partial R}{\partial u}$$

$$\frac{\partial X(u,v)}{\partial v} = -n\tilde{X}(u) \sin(nv) + n\tilde{X}(u) \cos(nv) + \frac{\partial R}{\partial v}$$

We observe that (3) and (4) are two basis vectors belonging to the tangent plane to surface with

$$\frac{\partial \tilde{X}(u)}{\partial u} = (c_1 ne^{nu} + c_2 ne^{nu} + c_2 ne^{nu} - c_3 ne^{nu} + c_4 e^{-nu} - c_4 ne^{nu})$$

Now we can build the metric

$$g_{11} = \langle \frac{\partial X(u,v)}{\partial u}, \frac{\partial X(u,v)}{\partial u} \rangle = E$$
\[ g_{12} = \langle \frac{\partial X(u, v)}{\partial u}, \frac{\partial X(u, v)}{\partial v} \rangle = g_{21} = \langle \frac{\partial X(u, v)}{\partial v}, \frac{\partial X(u, v)}{\partial u} \rangle = F \] (7)

\[ g_{22} = \langle \frac{\partial X(u, v)}{\partial v}, \frac{\partial X(u, v)}{\partial v} \rangle = G \] (8)

where the symbol \( <> \) is the scalar product. The line element of surface is

\[ ds^2 = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2 \] (9)

Now that we know the metric, we can calculate the surface area as following

\[
\iint_D \left| \frac{\partial X(u, v)}{\partial u} \wedge \frac{\partial X(u, v)}{\partial v} \right| dudv
\] (10)

where \( \wedge \) denotes the cross product, and the absolute value denotes the length of the vector.

If we want to characterize the facial surface by its curvature, we must construct the following Christoffel symbols of the first kind

\[ \Gamma_{ij\mu} = \frac{1}{2} \left( \frac{\partial g_{ij}}{\partial u^\mu} + \frac{\partial g_{i\mu}}{\partial u^j} - \frac{\partial g_{j\mu}}{\partial u^i} \right) \] (11)

obtaining

\[ \Gamma_{111} = \frac{1}{2} \frac{\partial g_{13}}{\partial u^1}; \Gamma_{121} = \Gamma_{112} = \frac{1}{2} \frac{\partial g_{13}}{\partial u^1}; \Gamma_{122} = \frac{\partial g_{14}}{\partial v} - \frac{1}{2} \frac{\partial g_{24}}{\partial u}; \Gamma_{211} = \frac{\partial g_{24}}{\partial u} - \frac{1}{2} \frac{\partial g_{14}}{\partial v}; \Gamma_{221} = \Gamma_{212} = \frac{1}{2} \frac{\partial g_{24}}{\partial v}; \Gamma_{222} = \frac{1}{2} \frac{\partial g_{24}}{\partial v} \]

By using the controvariant metric tensor, we can construct the Christoffel symbols of the first kind

\[ \Gamma^i_{kl} = \frac{1}{2} g^{im} \left( \frac{\partial g_{mk}}{\partial u^l} + \frac{\partial g_{ml}}{\partial u^k} - \frac{\partial g_{kl}}{\partial u^m} \right) \] (12)

and, thanks to the Egregium theorem, finally we get

\[ K = \frac{1}{G} \left[ \frac{\partial \Gamma^1_{22}}{\partial u} - \frac{\partial \Gamma^1_{12}}{\partial v} + \sum_{s=1}^{2} (\Gamma^s_{22} \Gamma^1_{1s} - \Gamma^s_{12} \Gamma^1_{2s}) \right] \] (13)
4 Conclusion

Elyan and Ugail have demonstrated how PDE approach could be used to efficiently represent and generate facial geometry of human faces through the use of real data of 3D facial scans. They have shown that with minimum input required for reconstruction of a given facial data set is the appropriate PDE boundary curves whereby the topology of the given face is preserved and represented accurately. Following this approach we have observed that in principle it is possible to recognize a person by a map of its Gaussian curvature.

References


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