Curve Fitting with Double-Exponential Equations

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Abstract
Illustrated is the interpolation of five or six evenly-spaced curvilinear data. The methods use the sum of two exponentials and a constant or a linear term and a constant. Both methods are based on the least-squares principle.

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1. Introduction
Recent papers in This Journal illustrate curvilinear interpolation by means of hyperbolas and exponentials [1,2,3]. For five and six equidistant, curvilinear data, the exponential interpolating forms are Eqs. (1) and (2), respectively.

\[ R = A + (B)C^x + (D)E^x \]  \hspace{1cm} (1)

\[ R = A + (B)x + (C)D^x + (E)F^x \]  \hspace{1cm} (2)
2. Five-point Method

The methods for the five- and six-point equidistant, curvilinear data are similar. Let the generating function \( y(x) \) be Eq. (3) and let it operate on the five numbers 1, 2, 3, 4, 5, respectively. The five (x,y) trial curvilinear data are thus (1,29), (3,51), (3,101), (4,219), (5,509), respectively. The sum of their squared deviations, denoted \( \sum \), is Eq. (4). This sum is based on Eq. (1).

\[
R = y(x) = (8)2^x + (1)3^x + 10
\]

\[
\]

The right-hand side of Eq. (4) is first differentiated with respect to \( A \) yielding Eq.(5). The symbol \( dA \) represents the partial derivative of \( \sum \) with respect to \( A \). Four additional equations are formed by differentiating \( \sum \) with respect to \( B, C, D, E \), respectively. The five-equation set is solved simultaneously for \( A, B, C, D, E \).

\[
dA = –1818+ 10A + 2(BC + DE + BC^2 + DE^2 + BC^3 + DE^3 + BC^4 + DE^4 + BC^5 + DE^5) = 0
\]

This process, using the cited data, renders 18 numerical solution sets. They can be tabulated for inspection. Sets that contain zeroes or identities (like \( D = D \)) are discarded. Sets that contain only real numbers are preferred if there is such a set. In the present case, one of the sets is \( \{A=10, B=8, C=2, D=1, E=3\} \). This method recovers the generating function, Eq. (3).

Now suppose \( y(x) = \{8.543, 31.76, 115.1, 331.3, 795.2\} \) for \( x = \{1, 2, 3, 4, 5\} \), respectively. There is presently no solution set containing only positive numbers. However, there is a solution set without zeroes or identities. For \( \{A,B,C,D,E\} \) it is \( \{21.94, -7.178 - 8.407I, 1.882 + 0.8097I, -7.178 + 8.407I, 1.882 - 0.8097I\} \), respectively. \( I \) is the imaginary unit. The corresponding interpolating equation is Eq. (6). Its coefficients are rounded and it reproduces the original data.

\[
R = 21.94 – (14.36)\exp(0.7170x)\cos(0.4064x) + (16.81)\exp(0.7170x)\sin(0.4064x)
\]

Double-exponential equations can be tried when easier forms like straight lines, parabolas, hyperbolas, and single-exponential equations are not satisfactory [1-3].
3. Six-point method

Let six curvilinear (x,y) data be (1,49), (2,81), (3,141), (4,269), (5,569), (6, 1321). The sum of their squared deviations is Eq. (7). It can be partially differentiated with respect to A,B,C,D,E,F thereby rendering six simultaneous equations. One member of those simultaneous equations is Eq. (8).

\[
\sum = (49 - A - B - CD - EF)^2 + (81 - A - 2B - CD^2 - EF)^2 \\
+ (141 - A - 3B - CD^3 - EF^3)^2 + (269 - A - 4B - CD^4 - EF^4)^2 \\
+ (569 - A - 5B - CD^5 - EF^5)^2 + (1321 - A - 6B - CD^6 - EF^6)^2 
\]  
(7)

\[
dA = -4860 + 2(CD + EF + CD^2 + EF^2 + CD^3 + EF^3 + CD^4 + EF^4 + CD^5 + EF^5 \\
+ CD^6 + EF^6) + 12A + 42B 
\]  
(8)

The six simultaneous equations generate eighteen numerical solution sets. An acceptable set is \{A=20, B=10, C=1, D=3, E=8, F=2\}. The equation representing them is therefore Eq. (9). It reproduces and interpolates the original data.

\[
R = 20 + 10x + (1)3^x + (8)2^x 
\]  
(9)

3. Discussion

The illustrated methods typically generate many sets of potential solutions to the simultaneous equations. A complete solution set contains no zeroes or identities. A complete set may contain complex numbers. Then the interpolating equation is expressed using sines and cosines. Such equations should reproduce the original data and interpolate real numbers using real numbers. If there is no complete solution set try another approach to generate an interpolating equation [1-3].

The reader may notice the similarity of the five-point method to Prony's original method: the sum of two exponentials plus a constant [4]. Prony did not suggest adding a linear term as in Eq. (2). That form seems to be new. It applies modern hardware and software that was not available to Prony.

The list below contains functions that are applied to [1,2,3,4,5] and [1,2,3,4,5,6], respectively, to generate five- and six-member data sets. They are the bases for five- and six-point interpolating equations. Each function \( F(x) \) is followed by the
five-point interpolating equation and then by the six-point interpolating equation.

\[
F(x) = \sinh(x/2)\cosh(x/3) + 10x + 20 \\
y(x) = 1480 - 1460(0.9930)^5 + 0.2766(2.271)^5 \\
y(x) = 19.97 + 10.09x - 0.2272(0.3949)^5 + 0.2515(2.300)^5
\]

\[
F(x) = 3(2^x) + \cosh(x) + x^3 + 100 \\
y(x) = 51.85 + 41.66(0.7882)^x + 12.36(1.930)^x \\
y(x) = 88.15 - 11.13x + 17.18(1.8354)^x + 0.0004852(6.351)^x
\]

\[
F(x) = (12)x^4 + (20)x^3 + (100)x + 10 \\
y(x) = 251.6 - (179.1)\exp(0.6896x)\cos(0.4089x) \\
+ (275.0)\exp(0.6896x)\sin(0.4089x) \\
y(x) = 8822 + 2666x - (8743)\exp(0.2655x)\cos(0.2581x) \\
- (669.4)\exp(0.2655x)\sin(0.2581x)
\]

\[
F(x) = (x^3)\ln((x+3)!) \\
y(x) = 34.66 - (29.02)\exp(0.7075x)\cos(0.4359x) \\
+ (25.55)\exp(0.7075x)\sin(0.4359x) \\
y(x) = 542.8 + 197.0x - (547.4)\exp(0.3533x)\cos(0.2932x) \\
+ (22.89)\exp(0.3533x)\sin(0.2932x)
\]

References

1. G. L. Silver, Hyperbola for curvilinear interpolation. Appl. Math. Sci. 7(30) (2013) 1477-1481 (Hikari Ltd). The text beneath Eq. (5) in this citation should read as follows: When linear Eq. (3) is solved for A, the result is Eq. (6).


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