Simulation of Asian Options Delta Using a Quadratic Congruential Pseudo-Random Numbers Generator

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Abstract
In this paper, we use the Monte Carlo simulation (MC) to calculate delta of an Asian option. Where delta measures the rate of change of option value (V) with respect to changes in the underlying asset’s price (S):

\[ \Delta = \frac{\partial V}{\partial S}. \]
Moreover, we use the quadratic congruential pseudorandom numbers generator, which have the following form:

\[ X_{n+1} = (aX_n^2 + bX_n + c) \mod m, \]

in order to evaluate its performance. We also compare results with two other chosen congruential generators.

**Keywords:** Asian options, Monte Carlo methods, Quadratic congruential pseudorandom numbers generator, Black and Scholes model, Delta, Linear congruential generator

## 1 Introduction

The Asian options depend on the path of asset prices and their payoff is based on the average of the underlying asset prices over some period previous to maturity, which makes them interesting as hedging instrument in markets with strong movements such as the exchange rate market. Several methods are used for pricing Asian options, we quote Geman and Yor (1993) who proposed numerical inversion of the Laplace transform of the Asian option price [8]. The method of "comonotonic approximations" has been developed by Kaas, Dhaene and Goovaerts (see Kaas et al. (2000)). An approach based on Taylor series approximations proposed by N.Ju [20]. For Monte Carlo simulation to price Asian options, we can name, for example, Kemna and Vorst (1990), or B.Lapeyre and E.Temam [3].

To hedge any short or long position on the market, one often resorts to the Greek calculation. In this sense, P. Boyle and A. Potapchik [25] proposed an efficient and complete survey about pricing and Greeks computation for Asian options.

The Monte Carlo (MC) simulation had much success in different fields due to its capacity of modeling, simplicity and the ability to deal with a large number of scenarios. It was introduced into options pricing by P.Boyle in (1977).

To apply (MC) method in finance, one often needs gaussian sequences, which can be computed from sequences of independent random variables uniformly distributed over the interval \([0, 1]\) by using The pseudo-random number generators (PRNGs).

Over the years, the PRNGs have interested many authors. Indeed, since 1949, D.H. Lehmer introduced the multiplicatif congruential generator, which became linear congruential generator thanks to Rotenberg (see[4]). The general form of this last is : \( X_{n+1} = (aX_n + c) \mod m \). Several results related in
this generator kind was done, either to improve their performance, or to study their properties (see [28], [21], [22], [23], [31], [33] and [34]).

Stochastic simulations performances are evaluated through two principal criteria, namely, the speed of execution and accuracy of results. The results accuracy of MC simulation in finance is closely related to the statistical properties of the congruential generator. In fact, The PRNGs must have statistical independence of the generated sequences, which are related to discrepancy of $k$-tuples of successive pseudorandom numbers [10]. The Equidistribution properties of quadratic congruential pseudorandom numbers are studied in [6]. To determinate a "goodness" of The generator and its compatibility to simulation methods many works was done (see [9], [10], [11], [12] and [13]).

The classical method for the generation of uniform pseudorandom numbers in the interval $[0, 1)$ is the linear congruential method. It is well known lattice structure makes this method useless for certain simulation purposes [6].

In this work, we present examples for which the linear generator is less "regular" comparing with the quadratic one. The body of this work will be organized as follows:

The second section and the third one are devoted to the presentation of the Black&Scholes model, with focusing on Asian options case (pricing and delta computation) and the MC method, respectively. In section 4, we expose the quadratic PRNG and its use in MC simulation. In section 5, we give numerical algorithm with related results. Finally, we conclude by some remarks and perspectives.

We note that throughout this work, we restrict our self to an Asian call option.

2 Financial Field

In this section we briefly describe Black&Scholes (B&S) model, we expose, in particular, case of Asian options.

2.1 B&S Model and Asian options

We are placed on a probabilized space $(\Omega; \mathcal{F}; \mathbb{P})$, The basic B&S model is considered with one risky asset, which evolves under the risk neutral measure denoted by $\mathbb{P}$, following The Stochastic differential equation (SDE): 

$$dS_t = S_t \left( r \, dt + \sigma \, dW_t \right) , \quad t \geq 0 ,$$

(1)

where $(W_t)_{t \geq 0}$ is a standard Brownian motion, which is built on the probabilized space and its natural filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. 

$\sigma \geq 0$ is the constant volatility, $r \geq 0$ is the constant risk-free rate [15], we
also consider $S_0 \geq 0$ as the initial asset price. The solution of SDE above is given \cite{5} by

$$S_t = S_0 \exp \left( (r - \frac{\sigma^2}{2}) t + \sigma W_t \right), \quad (2)$$

let $[0, T]$ be the time interval in which evolves our asset. We subdivide this interval into $N$ points $t_i$, which are equally-spaced. We are dealing with two types of Asian option:

- The geometric Asian option, which depends (in the discrete case) on the geometric average of the underlying asset values \cite{25}. Consequently, its payoff is given by

$$G_N = \left( \prod_{i=1}^{N} S_i \right)^{\frac{1}{N}}, \quad (3)$$

In the continuous case \cite{25}, the payoff can be rewritten as follows

$$G_T = \exp \left( \frac{1}{T} \int_0^T \ln(S_t) \, dt \right), \quad (4)$$

- The arithmetic Asian option, which relies (in the discrete case) on the arithmetic average of the underlying asset price \cite{25}. Hence, its payoff will be written

$$A_N = \frac{1}{N} \sum_{i=1}^{N} S_t, \quad (5)$$

The payoff of continuously sampled arithmetic Asian option is given by

$$A_T = \frac{1}{T} \int_0^T S_t \, dt, \quad (6)$$

In B&S model, under the risk-neutral probability, and for given strike $K \geq 0$ the price $C$ of these options is given \cite{14} by

$$C = \exp \left( -r (T - t) \right) \mathbb{E} \left[ \max(f(S_0, \cdots, S_T) - K ; 0) \right], \quad (7)$$

where $f(S_0, \cdots, S_T)$ is the payoff for a given option and $\mathbb{E}$ is the expectation under the risk-neutral probability. The price of continuously sampled Asian options can be estimated by taking sufficiently large values of $N$. However one must account for the inherent discretization bias resulting from the approximation of continuous time processes through discrete sampling as shown by Broadie et al. (1999).
2.2 Delta calculation

We focus on the basic approach to estimate price sensitivities, especially delta. In fact, we concentrate on our problem, which is computing delta of the B&S price of an Asian option call, where $\Delta = \frac{\partial C}{\partial S_0}$, with $C$ denotes the price in equation (7), and $S_0$ is the actual underlying asset price.

In the absence of a closed formula of this quantity, we will be brought to using MC simulation and we choose the naive method, which is the finite difference approximations. For direct methods, namely a pathwise (based on a technique generally called infinitesimal perturbation analysis), see Glasserman (1991). The likelihood ratio methods is introduced in [18]. For all other demarches the readers are referred to [25].

P. Boyle et al. had dealt with this kind of problems in [26]. For the forward difference estimation the idea is

1. With the equation (7), we calculate the terminal price $C(S_0)$ using a standard normal random variable $U$,

2. re-simulate this quantity with another standard normal random variable $U'$ to compute $C(S_0 + \epsilon)$, which is the Asian option price resulting from the perturbed initial price $(S_0 + \epsilon)$, where $U$ and $U'$ are independent.

Hence, our estimator for delta becomes

$$\tilde{\Delta} = \frac{C(S_0 + \epsilon) - C(S_0)}{\epsilon}.$$ 

With MC simulation, we can compute $N$ independent copies of $\tilde{\Delta}$. By using $N \to \infty$ the mean of these copies converges to the true finite-difference ratio, which is

$$\Delta = \frac{C(S_0 + \epsilon) - C(S_0)}{\epsilon}.$$ 

This estimator can be improved using the method of common random numbers, clearly, by using the same $U$ in the steps above. Even with the improvements in performance obtained from common random numbers, derivative estimate based on finite differences still suffer from two shortcomings: it is biased and they require multiple re-simulations [26]. However, we will see that this method give us enough-good results. further, we remark that sometimes, using common random numbers is less interesting than using independents ones.

While $\epsilon$ is arbitrarily taken, we realize that the variance of the estimator is very large (when $\epsilon$ is small). The reason for which authors considered the question. In this context P. W. Glynn had shown in [27] that for this method, the optimal rate of convergence is tipically $N^{-1/4}$. For more details concerning the choice of $\epsilon$, we invite the readers to consult [16].
Monte Carlo simulation

Thanks to its flexibility, the MC approach had success in finance. Two principal theorems build the main idea of this method, namely, the "Large Numbers Theorem" to estimate the expectation $E(Y_1)$ by the quantity $\lim_{n \to \infty} \frac{1}{N} \sum_{n=1}^{N} Y_n$ and the "Central Limit Theorem" for obtain confidence interval containing $E(Y_1)$ for a fixed probability, where $(Y_i)_{i \geq 1}$ is a sequence of independent identically distributed (i.i.d) real square-integrable random variables. The rate convergence of MC methods is typically $N^{-1/2}$.

Generally, MC method is composed of three capital steps:

1. simulate sample paths of the value which one wants to estimate ,
2. re-simulate $N$ independent reproductions of this value ,
3. take the average of all returns.

Let us suppose that $[0,T]$ is the time interval in which evolves our underlying asset, we will assume that $t_i = ih$, with $h = \frac{T}{N}$. Thus the MC simulation version of equation (2) is given by

$$S_i = S_{i-1} \exp \left( \mu h + \sigma U \sqrt{h} \right), \quad i = 1 \cdots N ,$$

where $\mu = r - \frac{\sigma^2}{2}$, $S_i$ denotes the underlying asset price at date $t_i$, and $U$ is a random variable taken from the standard normal distribution.

From (8), we realize clearly need of source of random variables following standard normal distribution. In what follows, we expose the quadratic PRNG, which will be used in our simulation.

The quadratic PRNG

The subject of the congruential generators was dealt by Knuth (1968) in the encyclopedic work (see [4]). In this latter, he exposed various forms of PRNG, in particular, the linear and quadratic ones. Generally, the PRNGs are digital sequences defined in $\mathbb{Z}_n$. They are characterized by their modulus $m$, their seed (initial value $Y_0$), and their forms.

According to [9], let $\mathbb{Z}_n = \{0, 1, ..., n-1\}$, $m = p^l$, with $p$ is a prime number, and $l \geq 2$ is an integer. For integers $a, b, c, Y_0$, a quadratic congruential sequence $(Y_n)_{n \geq 0}$ of elements of $\mathbb{Z}_m$ is defined by

$$Y_{n+1} = (aY_n^2 + bY_n + c) \mod m , \quad n \geq 0 .$$

(9)
With chosen $Y_0$, this form generate a sequence of random variables in the interval $[0, m)$, and to obtain $(X_n)_{n \geq 0}$ the sequence of uniform random numbers in the interval $[0, 1)$, we normalize $Y_n$ by dividing by $m$. Clearly, $X_n = Y_n/m$ will be a sequence of uniform random numbers in the interval $[0, 1)$.

It is known [9] that a quadratic congruential sequence with $p \geq 5$ has maximal period length $m$ if and only if

$$
\begin{align*}
\{ & a \equiv 0 \pmod{p}, \\
& b \equiv 1 \pmod{p}, \\
& c \not\equiv 0 \pmod{p}.
\end{align*}
$$

5 Numerical Algorithm and Results

In the preceding sections we had exposed our work tools. In this one these tools will be implemented. We start initially with given the algorithm.

5.1 Numerical Algorithm

In addition to the quadratic PRNG, we choose two congruential random numbers generators, which have the form $X_{n+1} = (aX_n + c) \mod m$.

- The first one is $LCG(69069, 1, 2^{32})$, with parameters

  $$
a = 69069, \quad c = 1, \quad \text{and} \quad m = 2^{32}.
$$

- The second is a multiplicative linear congruential generator, it has been used in IBM computers, called ”standard minimal” in [32], where

  $$
a = 16807, \quad c = 0, \quad m = 2^{31} - 1.
$$

As shown above, we first discretize the time interval $[0, T]$ into $N$ parts which length is equal to $h = \frac{T}{N}$, where $t_i = ih$. So we have the time’s loop ($i = 0, 1, \cdots, N - 1$). A MC loop will be ($j = 0, 1, \cdots, M - 1$).

The generators of random numbers produce uniform pseudo-random variate $U(0, 1)$. In order to extract a reduced centered normal law variate $\mathcal{N}(0, 1)$, which will be used in the equation (8), we call on the Box-Muller method (see [7]). In fact, Suppose $X_1$ and $X_2$ are independent random variables that are uniformly distributed in the interval $(0, 1)$, consider the random variables

$$
U_1 = \sqrt{-2\ln X_1} \cos(2\pi X_2), \\
U_2 = \sqrt{-2\ln X_1} \sin(2\pi X_2).
$$
Then \((U_1, U_2)\) will be a pair of independent random variables from the same normal distribution with mean zero, and unit variance \([7]\). Now our proposed algorithm to compute an Asian call option delta is as follows:

In loops
\[
\text{For } j = 0, \cdots, M - 1, \\
\text{For } i = 0, \cdots, N - 1,
\]

1. Divide the time interval in \(N\) step which lengths are equal to \(h = T/N\),

2. Initialize two seeds,

3. Generate \(X_1(i, j)\) and \(X_2(i, j)\) by the quadratic (resp.Linear) PRNG,

4. Generate \(U_1(i, j)\) and \(U_2(i, j)\) the two random variables following a standard normal distribution,

5. Use \(U_1(i, j)\) to calculate the trajectory \((S_t)_{0 \leq t \leq T}\) of the underlying asset price at points \(t_i\) by equation (8),

6. Use \(U_2(i, j)\) to calculate the trajectory \((S'_t)_{0 \leq t \leq T}\) of the perturbed underlying asset price at the value \(S_0 + \epsilon\) (section 2.2) at points \(t_i\) by equation (8),

7. Calculate the average of both trajectories to deduce prices using equation (7),

8. For each iteration of MC loop, calculate the corresponding delta, using the estimator \(\Delta = \frac{C(S_0 + \epsilon) - C(S_0)}{\epsilon}\),

9. Make the average of all computed delta to have the final estimation of delta.

### 5.2 Numerical Results

For efficiency, and speed of execution we implemented all of the Monte Carlo methods in C. Concerning the quadratic PRNG used in this work, we had taken a basic one, with the following parameters

\[ a = 101, \quad b = 203, \quad c = 103 \quad \text{and} \quad m = 10^6. \]

In the following tables, we consider Asian option with parameters

\[ K = 100, \quad S_0 = 100, \quad r = 0.05, \quad \text{and} \quad T = 1. \]

All estimates are based on 1000 readings and 100 000 replications of sample paths. Standard errors and execution time (in seconds) are also given. In the table1, \(FD^1\) denotes Finite difference method used in \([25]\), \(FD^Q\) is finite difference method with a quadratic PRNG, \(FD^{LCG}\) is the same method with \(LCG(69069, 1, 2^{32})\), and \(FD^{ST, min}\) denotes finite difference method with ”standard minimal”.

Except the \(LCG(69069, 1, 2^{32})\), which gives a value far from the desired value for \(\sigma = 0.5\), the quadratic PRNG and the ”standard minimal” one produces satisfactory results concerning delta of the Arithmetic Asian call
Table 1: Delta of an Arithmetic Asian call option

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\sigma = 0.1$</th>
<th>$\sigma = 0.3$</th>
<th>$\sigma = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Delta</td>
<td>Exec_Time</td>
<td>Delta</td>
</tr>
<tr>
<td>$FD^1$</td>
<td>0.6599(0.0016)</td>
<td>-</td>
<td>0.5668(0.0021)</td>
</tr>
<tr>
<td>$FD^Q$</td>
<td>0.6590(0.0026)</td>
<td>49</td>
<td>0.5794(0.0022)</td>
</tr>
<tr>
<td>$FD^{LCG}$</td>
<td>0.6544(0.0026)</td>
<td>45</td>
<td>0.5794(0.0022)</td>
</tr>
<tr>
<td>$FD^{ST\text{min}}$</td>
<td>0.6609(0.0025)</td>
<td>48</td>
<td>0.5711(0.0022)</td>
</tr>
</tbody>
</table>

Table 2: Delta of a Geometric Asian call option

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\sigma = 0.1$</th>
<th>$\sigma = 0.3$</th>
<th>$\sigma = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Delta</td>
<td>Exec_Time</td>
<td>Delta</td>
</tr>
<tr>
<td>Exact value</td>
<td>0.6556</td>
<td>-</td>
<td>0.5379</td>
</tr>
<tr>
<td>$FD^1$</td>
<td>0.6578(0.0016)</td>
<td>-</td>
<td>0.5341(0.0020)</td>
</tr>
<tr>
<td>$FD^Q$</td>
<td>0.6568(0.0026)</td>
<td>68</td>
<td>0.5352(0.0043)</td>
</tr>
<tr>
<td>$FD^{LCG}$</td>
<td>0.6520(0.0026)</td>
<td>67</td>
<td>0.5163(0.0042)</td>
</tr>
<tr>
<td>$FD^{ST\text{min}}$</td>
<td>0.6587(0.0025)</td>
<td>67</td>
<td>0.5350(0.0041)</td>
</tr>
</tbody>
</table>

In the following table, ”Exact value” denotes the value presented in [25] computed with analytical method, for the other notations, they are taken the same as in the table above.

For all values of $\sigma$ the quadratic PRNG has a good behavior, consequently, it gives good results. Moreover, we can see that for $\sigma = 0.1$ and $\sigma = 0.3$, the results arising from the quadratic PRNG are nearer, to the Exact value, than others given in [25] computed by the same method While the two other linear ones had an irregular behavior depending on value of $\sigma$.

we present below graphs which correspond to standard gaussian density resulting from the three PRNG, and using box-muller method. They are compared with reduced centered normal law generated by the R statistical software (dotted lines).

The sample considered corresponds to 10000 random variables. From left to right, the graphs correspond, respectively, to results of the ”standard minimal” PRNG, the quadratic PRNG and the LCG presented above.
6 Conclusion

In this work, we saw the importance of the generator choice in MC simulation, we also shown that The quadratic pseudo-random numbers generator is efficient to simulate the Asian option delta, either for the geometric or the arithmetic case. Further, we remark that the chosen linear generators are not compatible such simulations.

However, one can improve the results arising from The quadratic PRNG by using a more efficient PRNG in MC simulation.

For ameliorating the MC simulation quality, we have to reduce the execution time. In this sense, in the future works, we will utilize a parallel algorithm. also, we are near from achieve a work on a new, more efficient quadratic generator.

References


Simulation of Asian options delta


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