A Note on the New Approach to
$q$-Bernoulli Polynomials

J. Seo

Department of Applied Mathematics
Pukyung National University
Busan, 608-737, Republic of Korea

S. H. Lee

Division of General Education
Kwangwoon University
Seoul 139-701, Republic of Korea
leesh58@kw.ac.kr

T. Kim

Division of General Education
Kwangwoon University
Seoul 139-701, Republic of Korea
kwangwoonmath@hanmail.net

S.-H. Rim

Department of Mathematics Education
Kyungpook National University
Taegu 702-701, Republic of Korea
shrim@knu.ac.kr

Abstract. In this paper, we construct new $q$-extension of Bernoulli polynomials. These $q$-Bernoulli polynomials are useful to study various identities of Carlitz’s $q$-Bernoulli numbers.
1. Introduction

Throughout this paper \( \mathbb{Z}_p, \mathbb{Q}_p \) and \( \mathbb{C}_p \) will respectively denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers and the completion of algebraic closure of \( \mathbb{Q}_p \). Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu_p(p)} = \frac{1}{p} \).

In this paper, we assume that \( q \in \mathbb{C}_p \) with \( |1 - q|_p < p^{-\frac{1}{p-1}} \). So that \( q^x = \exp(x \log q) \) for \( x \in \mathbb{Z}_p \). The \( q \)-number of \( x \) is denoted by \( [x]_q = \frac{1 - q^x}{1 - q} \). Note that \( \lim_{q \to 1} [x]_q = x \).

Let \( d \) be a fixed integer bigger than 0 and let \( p \) be a fixed prime number. We set
\[
X_d = \lim_{N \to \infty} \mathbb{Z} / dp^N \mathbb{Z}, \quad X^* = \bigcup_{0 < a < dp} (a + dp \mathbb{Z}_p),
\]
\[
a + dp^N \mathbb{Z}_p = \{ x \in X | x \equiv a \pmod{dp^N} \},
\]
where \( a \in \mathbb{Z} \) lies in \( 0 \leq a < dp \), (see [1-17]).

Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable functions on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by
\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad ( \text{see [10,12]} ).
\]

As is well known, Bernoulli polynomials are defined by the generating function to be
\[
\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad ( \text{see [1 - 12]} ).
\]
When \( x = 0 \), \( B_n = B_n(0) \) are called the \( n \)-th Bernoulli numbers. From (1.2), we note that
\[
B_n(x) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) B_k x^{n-k} = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) B_{n-k} x^k.
\]
In [3, 4], L. Carlitz defined the \( q \)-extension of Bernoulli numbers as follows:
\[
\beta_{0,q} = 1, q(q\beta + 1)^k - \beta_{k,q} = \delta_{k,1}
\]
with the usual convention of replacing \( \beta_q^l \) by \( \beta_{l,q} \).

From (1.4), we note that
\[
\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) (-1)^l \frac{l+1}{[l+1]_q}, \quad (n \geq 0).
\]
In [10], it was known that properties of those polynomials. In this paper, we give a new approach of \( q \)-extension of Bernoulli polynomials as follows:

\[
\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]^{n-l} q^l x \beta_{i,q}, \quad \text{(see [3,4])}. \tag{1.6}
\]

In [10], it was known that

\[
\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n \, d\mu_q(x), \quad \beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n \, d\mu_q(x).
\]

In this paper, we give a new \( q \)-extension of Bernoulli polynomials which are different Carlitz’s \( q \)-Bernoulli polynomials. Finally, we investigate several properties of those polynomials.

2. A NEW APPROACH OF \( q \)-BERNOULLI POLYNOMIALS

Let us consider the following modified \( q \)-Bernoulli polynomials which are defined by the generation function to be

\[
\sum_{n=0}^{\infty} \tilde{\beta}_{n,q}(x)(q-1)^n \frac{t^n}{n!} = e^{((q-1)x-1)t} \left( \sum_{i=0}^{n} \frac{i+1}{[i+1]_q} \frac{t^i}{i!} \right). \tag{2.1}
\]

From (2.1), we can derive the following equation:

\[
\sum_{n=0}^{\infty} (q-1)^n \tilde{\beta}_{n,q}(x) \frac{t^n}{n!} = \left( \sum_{l=0}^{\infty} \frac{((q-1)x-1)^l}{l!} t^l \right) \left( \sum_{i=0}^{\infty} \frac{i+1}{[i+1]_q} \frac{t^i}{i!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} \frac{n!((q-1)x-1)^{n-i}}{(n-i)!} \frac{i+1}{[i+1]_q} \right) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} \binom{n}{i} ((q-1)x-1)^{n-i} \frac{i+1}{[i+1]_q} \right) \frac{t^n}{n!}. \tag{2.2}
\]

Thus, by (2.2), we get

\[
\tilde{\beta}_{n,q}(x) = \frac{1}{(q-1)^n} \sum_{i=0}^{n} \binom{n}{i} \frac{i+1}{[i+1]_q} ((q-1)x-1)^{n-i} \tag{2.3}
\]

\[
= \frac{1}{(1-q)^n} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-1}{j} \frac{i+1}{[i+1]_q} \frac{(-1)^{i+j}(q-1)^j x^j}{n!}.
\]

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.1. For \( n \geq 0 \), we have

\[
\tilde{\beta}_{n,q}(x) = \frac{1}{(q-1)^n} \sum_{i=0}^{n} \binom{n}{i} \frac{i+1}{[i+1]_q} ((q-1)x-1)^{n-i}
\]

\[
= \frac{1}{(1-q)^n} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-1}{j} \frac{i+1}{[i+1]_q} \frac{(-1)^{i+j}(q-1)^j x^j}{n!}.
\]
From (1.1) and (1.5), we can derive

$$\int_{\mathbb{Z}_p} e^{(x+[y]_q)(q-1)t} d\mu_q(y) = e^{((q-1)x-1)t} \left( \sum_{i=0}^{\infty} \frac{i + 1}{[i + 1] q^i} t^i \right)$$

$$= \sum_{n=0}^{\infty} (q - 1)^n \tilde{\beta}_{n,q}(x) \frac{t^n}{n!},$$

(2.4)

and

$$\int_{\mathbb{Z}_p} e^{(x+[y]_q)(q-1)t} d\mu_q(y) = \sum_{n=0}^{\infty} (q - 1)^n \int_{\mathbb{Z}_p} (x + [y]_q)^n d\mu_q(y) \frac{t^n}{n!}.$$  

(2.5)

Therefore, by (2.4) and (2.5), we obtain the following theorem.

**Theorem 2.2.** For $n \geq 0$, we have

$$\int_{\mathbb{Z}_p} (x + [y]_q)^n d\mu_q(y) = \tilde{\beta}_{n,q}(y).$$

When $x = 0$, $\tilde{\beta}_{n,q} = \tilde{\beta}_{n,q} = \beta_{n,q}$ are $n$-th Carlitz $q$-Bernoulli numbers. Now, we observe that

$$\int_{\mathbb{Z}_p} (x + [y]_q)^n d\mu_q(y) = \sum_{l=0}^{\infty} \binom{n}{l} x^l \int_{\mathbb{Z}_p} [y]_q^{n-l} d\mu_q(y)$$

$$= \sum_{l=0}^{n} \binom{n}{l} x^l \beta_{n-l,q}$$

(2.6)

where $\beta_{n,q}$ are the $n$-th Carlitz's $q$-Bernoulli numbers. Therefore, by (2.6), we obtain the following corollary.

**Corollary 2.3.** For $n \geq 0$, we have

$$\tilde{\beta}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^l \beta_{n-l,q} = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,q} x^{n-l},$$

where $\beta_{n,q}$ are the Carlitz's $q$-Bernoulli numbers.

Note that

$$\tilde{\beta}_{n,q}(x) = \int_{\mathbb{X}} (x + [y]_q)^n d\mu_q(y)$$

$$= [d]_q^{n-1} \sum_{a=0}^{d-1} q^{a(n+1)} \int_{\mathbb{Z}_p} \left( \frac{x + [a]_q}{q^a [d]_q} + [y]_q^a \right)^n d\mu_q(x)$$

$$= [d]_q^{n-1} \sum_{a=0}^{d-1} q^{a(n+1)} \tilde{\beta}_{n,q} \left( \frac{x + [a]_q}{q^a [d]_q} \right).$$

(2.7)

Therefore, by (2.7), we obtain the following theorem.
Theorem 2.4. For \( d \in \mathbb{N}, n \geq 0 \), we have

\[
\tilde{\beta}_{n,q}(x) = [d]_q^{n-1} \sum_{a=0}^{d-1} q^{a(n+1)} \tilde{\beta}_{n,q,a} \left( \frac{x + [a]_q}{q^a [d]_q} \right).
\]

Let \( \tilde{\beta}(t) \) be the generating function of Carlitz’s \( q \)-Bernoulli numbers:

\[
\tilde{\beta}(t) = \sum_{n=0}^\infty \frac{\beta_{n,q} t^n}{n!} = e^{-t} \sum_{i=0}^\infty \frac{i+1}{[i+1]_q !} t^i.
\]

Then, we have

\[
\tilde{\beta} \left( \frac{\partial}{\partial x} \right)^k x^k = \sum_{n=0}^\infty \frac{\beta_{n,q} n!}{n!} \left( \frac{\partial}{\partial x} \right)^n x^k
\]

\[
= \sum_{n=0}^{k} \frac{(k)_n}{n!} \beta_{n,q} x^{n-k}, \quad (k \geq 0),
\]

where \((k)_n = k(k-1)(k-2) \cdots (k-n+1)\). By (2.9), we get

\[
\tilde{\beta} \left( \frac{\partial}{\partial x} \right)^k x^k = \sum_{n=0}^{k} \frac{(k)_n}{n!} \beta_{k,n} \beta_{n,q} x^{n-k} = \beta_{k,q}(x).
\]

Now, we consider

\[
\tilde{\beta} \left( \frac{\partial}{\partial x} \right) e^{tx} = \sum_{n=0}^\infty \frac{\beta_{n,q} n!}{n!} \left( \frac{\partial}{\partial x} \right)^n e^{tx}
\]

\[
= \sum_{n=0}^\infty \frac{\beta_{n,q} t^n}{n!} e^{tx}
\]

\[
= \left( e^{-t} \sum_{i=0}^\infty \frac{i+1}{[i+1]_q !} t^i \right) e^{tx}
\]

\[
= e^{(x-1)t} \sum_{i=0}^\infty \frac{i+1}{[i+1]_q !} t^i
\]

\[
= \beta(x : t).
\]

REFERENCES


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