Eigenvalue Problems for Crosswise Vibration Equations of Beam with Continuous External Force

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Abstract

The eigenvalue and its related problems for crosswise vibration equation of the beam with continuous external force, which comes from the vibration phenomena, are studied. The existence and monotonicity of the eigenvalues are proved. In additional, we also obtain the orthogonality and completeness of the corresponding eigenfunctions.

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1 Introduction

Recently, there has been increasing interest in the eigenvalue problems with fixed boundaries, which is mainly motivated by the vibration phenomena of the homogeneous beams\([1],[2],[3]\). In fact, it is difficult to obtain the actual values of the eigenvalues (i.e. the natural frequency) in general case, and we only can achieve its estimates or approximations. However, the natural frequencies of the beams must not be close to the external force frequencies in designing load-carrying member, and taking into account resonance factors in the engineering\(\text{(Otherwise, this would cause resonance, and has great collapsing power)}\). It is interesting in studying this problems both in differential equations theory and mechanics and physics applications\([1],[4]\). On the other hand, if the beam is attached continuous external force, the vibration laws satisfy the following integro-differential equation\([4]\)

\[-(p(x)y'(x))' + q(x)y(x) + \int_a^b G(x, t)y(t)dt = \lambda y(x), x \in (a, b), \]

\[y(a) = y(b) = 0,\]

where \((a, b) \subset R\) is a bounded open interval, \(p(x), q(x), G(x, t)\) are satisfied:

(i) \(p(x), q(x) \in C^1(a, b) \cap C[a, b]\), and \(\min_{x \in [a,b]} p(x) = p_0 > 0, \min_{x \in [a,b]} q(x) = q_0 > 0\).

(ii) \(G(x, t) \in C([a, b] \times [a, b]).\)

In this paper, we study the properties of the eigenvalues and eigenfunctions for the problems (1) and (2). This paper is organized as follows. In Section 2, we discuss the existence and monotonicity of the eigenvalues. Section 3 is devoted to the orthogonality and completeness of the corresponding eigenfunctions.

2 The Properties of eigenvalues

In this section, we study the eigenvalues and their related problems for integro-differential (1)-(2), and obtain the existence and monotonicity of the
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In the sequel, we only study the related problems for \( G(s, t) = g(s)g(t), g(s) \in C[a, b] \).

Let \( L^2[a, b] \) be the square-integrable function space on \([a, b] \), and the inner product is defined by

\[
(y_1, y_2)_L \equiv \int_a^b y_1(x)y_2(x)dx, \quad \forall y_1, y_2 \in L^2[a, b].
\]  

(6)

In addition, for \( y_1, y_2 \in C^1_0[a, b] \), the inner product is defined by

\[
(y_1, y_2)_H \equiv \int_a^b \left[ p(x)y'_1(x)y'_2(x)dx + q(x)y_1(x)y_2(x) + y_2(x)\int_a^b G(x, t)y_1(t)dt \right] dx,
\]

\[ \forall y_1, y_2 \in C^1_0[a, b], \]  

(7)

where \( p(x), q(x) \in C[a, b] \), and there exist \( p_0 > 0, q_0 > 0 \) such that \( p(x) > p_0, q(x) > q_0 \) for each \( x \in [a, b] \). The norm induced by (7) is denoted by \( \| \cdot \|_H \), and the space \( H^{0,1}_{p,q,g}[a, b] \) is the closure of \( C^1_0[a, b] \) in \( L^2[a, b] \). It is easy to prove that \( L^2[a, b] \) and \( H^{0,1}_{p,q,g}[a, b] \) are Hilbert spaces[5].

**Definition 2.1** Let \( \lambda \) be a real number. We call that \( y \) is a weak solution of (1)-(2) in \([a, b]\) if \( y \in H^{0,1}_{p,q,g}[a, b] \) and

\[
(y, \eta)_H = \lambda(y, \eta)_L,
\]  

(8)

for all functions \( \eta \in C^1_0[a, b] \).

To prove the existence of the weak solutions, we consider the variational problem as follows

\[
J(y) = \frac{(y, y)_H}{(y, y)_L}, \quad \forall y \in H^{0,1}_{p,q,g}[a, b].
\]  

(9)

The following Lemmas will be essential for our main results:

**Lemma 2.2** Assume \( 0 \neq y_0 \in H^{0,1}_{p,q,g}[a, b] \), and \( y_0 \) satisfies

\[
J(y_0) = \inf_{y \in H^{0,1}_{p,q,g}[a, b]} J(y),
\]

then \( y_0 \) is a weak solution of the problem (1)-(2). Furthermore, we have \( \lambda = \frac{(y_0, y_0)_H}{(y_0, y_0)_L} \).

**Proof** For every \( \eta \in C^1_0[a, b] \) and real number \( \alpha \), we set

\[
M \equiv (y_0 + \alpha \eta, y_0 + \alpha \eta)_H = (y_0, y_0)_H + 2\alpha(y_0, \eta)_H + \alpha^2(\eta, \eta)_H,
\]

\[
N \equiv (y_0 + \alpha \eta, y_0 + \alpha \eta)_L = (y_0, y_0)_L + 2\alpha(y_0, \eta)_L + \alpha^2(\eta, \eta)_L.
\]
Since \( J[\alpha] \equiv J(y_0 + \alpha \eta) \) achieves a minimum value at \( \alpha = 0 \), we have \( J'[0] = 0 \). i.e.
\[
\frac{d}{d\alpha} \left( \frac{M}{N} \right) \bigg|_{\alpha=0} = 0.
\]
By simply calculation, we get
\[
(y_0, \eta)_H - \frac{(y_0, y_0)_H}{(y_0, y_0)_L} (y_0, \eta)_L = 0. \quad (10)
\]
Taking \( \lambda = \frac{(y_0, y_0)_H}{(y_0, y_0)_L} \) in (10), we have \( (y_0, \eta)_H = \lambda (y_0, \eta)_L \). So \( y_0 \) is a weak solution of the problem (8).

**Lemma 2.3** The bounded sets in \( H^{0,1}_{p,q,g}[a, b] \) are compact sets in \( C[a, b] \).

**Proof** Let \( \{y_n\} \) be a sequence in \( H^{0,1}_{p,q,g}[a, b] \) with \( ||y_n||_H \leq Y \). Since \( y_n \in H^{0,1}_{p,q,g}[a, b] \), then \( y_n \) has a weak derivative \( y'_n \), and
\[
y_n(x) = y_n(a) + \int_a^x y'_n(x)dx = \int_a^x y'_n(x)dx.
\]
Hence, by H"older inequality, we have
\[
|y_n(x)| \leq \int_a^x |y'_n(x)|dx \\
\leq \left( \int_a^x |y'_n(x)|^2dx \right)^{1/2} \left( \int_a^x dx \right)^{1/2} \\
\leq \sqrt{\frac{b-a}{p_0}} \left( \int_a^x p(x)|y'_n(x)|^2dx \right)^{1/2} \\
\leq \sqrt{\frac{b-a}{p_0}} \left( \int_a^b p(x)|y'_n(x)|^2dx + \int_a^b q(x)|y_n(x)|^2dx \\
+ \int_a^b \int_a^b G(x, t)y_n(x)y_n(t)dxdt \right)^{1/2} \\
\leq \sqrt{\frac{b-a}{p_0}} Y, n = 1, 2, \cdots .
\]
We get that the sequence \( y_n(x)(n = 1, 2, \cdots ) \) is uniformly bounded in \( C[a, b] \). On the other hand, for real number \( h \) and \( x + h \in [a, b] \), by using the method stated above, we have
\[
|y_n(x + h) - y_n(x)| \leq \int_x^{x+h} |y'_n(x)|dx \leq \sqrt{\frac{h}{p_0}||y_n||_H} \leq \sqrt{\frac{h}{p_0}Y}.
\]
Therefore \( y_n(x)(n = 1, 2, \cdots) \) are equi-continuous in \( C[a, b] \). By using the Arzela-Ascoli Theorem\(^6\), we obtain that there is a subsequence \( \{y_{n_k}\} \subset \{y_n\} \) such that \( \{y_{n_k}\} \) is convergent in \( C[a, b] \). i.e. \( \{y_n\} \) is a compact set in \( C[a, b] \).

**Theorem 2.4** (Existence of minimum function) There exists \( y \in H_{p,q,g}^{0,1}[a, b] \) such that 

\[
J(y) = \inf_{z \in H_{p,q,g}^{0,1}[a, b]} J(z).
\]

**Proof** Without loss of generality, we only need to show that there exists \( y \in H_{p,q,g}^{0,1}[a, b] \), such that \( (y, y)_H = \inf_{z \in H_{p,q,g}^{0,1}[a, b]} (z, z)_H \), where \( z \in H_{p,q,g}^{0,1}[a, b] \), and \( (z, z)_L = 1 \).

Let \( \lambda = \inf_{z \in H_{p,q,g}^{0,1}[a, b]} (z, z)_H \). Since \( (z, z)_L = 1 \), we have

\[
(z, z)_H = ||z||_H^2 = \int_a^b p(x)(z')^2 dx + \int_a^b q(x)z^2 dx + \int_a^b \int_a^b G(x, t)z(x)z(t) dt dx \\
\geq \int_a^b p(x)(z')^2 dx + q_0 \int_a^b z^2 dx + \left( \int_a^b g(x)z(x) dx \right)^2 > q_0 > 0.
\]

Hence \( \lambda > 0 \). In the following, we shall prove there exists \( y \in H_{p,q,g}^{0,1}[a, b] \) with \( (y, y)_H = \lambda \). Let

\[
K = \{ z \in H_{p,q,g}^{0,1}[a, b] | (z, z)_L = 1 \}.
\]

Taking the minimizing sequence \( \{y_k\} \subset K \), i.e.

\[
||y_k||_H^2 = (y_k, y_k)_H \to \lambda, \quad (k \to \infty), \quad (11)
\]

and using the parallelogram formula\(^5\)

\[
||y_k + y_l||_H^2 + ||y_k - y_l||_H^2 = 2||y_k||_H^2 + 2||y_l||_H^2. \quad (12)
\]

We have: for \( \forall \epsilon > 0 \), as \( k, l \) are sufficiently large

\[
||y_k||_H^2 < \lambda + \frac{\epsilon}{4}, \quad ||y_l||_H^2 < \lambda + \frac{\epsilon}{4}.
\]

Therefore

\[
2||y_k||_H^2 + 2||y_l||_H^2 < 4\lambda + \epsilon \quad (13)
\]

On the other hand

\[
||y_k + y_l||_H^2 \geq \lambda ||y_k + y_l||_L^2 \quad (14)
\]

Substituting (14) and (13) into (12), we have

\[
||y_k - y_l||_H^2 \leq 4\lambda + \epsilon - \lambda ||y_k + y_l||_L^2.
\]
By (12), we get

\[ ||y_k - y_l||_H^2 \leq 4\lambda + \epsilon - \lambda(2||y_k||_L^2 + 2||y_l||_L^2 - ||y_k - y_l||_H^2) \]

\[ = \epsilon + \lambda||y_k - y_l||_L^2. \] (15)

Since \( \{y_k\} \) is a bounded sequence in \( H_{p,q,g}^{0,1}[a,b] \) and \( H_{p,q,g}^{0,1}[a,b] \) is compact in \( C[a,b] \), this implies that there exists a subsequence \( \{y_{k_i}\} \), such that

\[ \max_{[a,b]} |y_{k_i}(x) - y_{k_j}(x)| \to 0, \quad (i, j \to \infty). \]

i.e.

\[ ||y_{k_i} - y_{k_j}||_L^2 = \int_a^b (y_{k_i} - y_{k_j})^2 \, dx \leq (b - a) \max_{[a,b]} |y_{k_i} - y_{k_j}|^2 \to 0(i, j \to \infty). \]

Hence, if \( i, j \) are sufficiently large, then \( ||y_{k_i} - y_{k_j}||_L^2 \leq \epsilon/\lambda \). That is to say, if \( i, j \) are sufficiently large, from (14), we have

\[ ||y_{k_i} - y_{k_j}||_H^2 \to 0(i, j \to \infty). \]

Let \( \Omega_1 = H_{p,q,g}^{0,1}[a,b] \). We call that

\[ \lambda_1 = \inf_{y \in \Omega_1, y \neq 0} \frac{(y, y)_H}{(y, y)_L} \]

is the first eigenvalue of problem (1)-(2). We call that \( y_1 \) is the corresponding eigenfunction of the first eigenvalue \( \lambda_1 \) if \( y_1 \neq 0, y_1 \in K \), such that

\[ \frac{(y_1, y_1)_H}{(y_1, y_1)_L} = \lambda_1. \]

By Theorem 2.4, the eigenfunction \( y_1 \) is exist.

Let

\[ \Omega_2 = \{y \in K | (y, y)_L = 0\}. \]

As in the proof of Theorem 2.4, we obtain that there exists \( y_2 \in \Omega_2(y_2 \neq 0) \) such that

\[ \lambda_2 = \frac{(y_2, y_2)_H}{(y_2, y_2)_L} = \min_{y \in \Omega_2, y \neq 0} \frac{(y, y)_H}{(y, y)_L}. \]
In general case, if
\[ \Omega_n = \{ y \in K | (y, y_1)_L = (y, y_2)_L = \cdots = (y, y_{n-1})_L = 0 \}, \]
we obtain that there exists \( y_n \in \Omega_n (y_n \neq 0) \) such that
\[ \lambda_n = \frac{(y_n, y_n)_H}{(y_n, y_n)_L} = \min_{y \in \Omega_n, y \neq 0} \frac{(y, y)_H}{(y, y)_L}. \]

**Definition 2.6** The real number \( \lambda_n (n = 1, 2, \cdots) \) is called the \( n \)th eigenvalue of problem (1)-(2). The function \( y_n (n = 1, 2, \cdots) \) is called the corresponding eigenfunction of the \( n \)th eigenvalue \( \lambda_n \).

**Theorem 2.7** Let \( \lambda_n (n = 1, 2, \cdots) \) be \( n \)th eigenvalue of the problem (1)-(2). Then we have
\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots. \]

**Proof** From the definition of \( \Omega_n \), we get that
\[ \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \cdots. \]
This implies
\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots. \]

3 The Properties of eigenfunctions

In this section, we will study the properties of the eigenfunctions and their related problems for the integro-differential (1)-(2).

**Theorem 3.1** Let \( y_1(x) \) and \( y_2(x) \) be the corresponding eigenfunctions of (1)-(2) of the eigenvalues \( \lambda_1 \) and \( \lambda_2 (\lambda_1 \neq \lambda_2) \), respectively. Then, \( y_1(x) \) and \( y_2(x) \) are orthogonal in \([a, b]\), i.e. \( \int_a^b y_1(x)y_2(x)dx = 0 \).

**Proof** Since \( y_1(x) \) and \( y_2(x) \) are the corresponding eigenfunctions of (1)-(2) of the eigenvalues \( \lambda_1 \) and \( \lambda_2 \), respectively. Substituting \( y_1(x), \lambda_1 \) and \( y_2(x), \lambda_2 \) into (1), we have
\[ -(p(x)y'_1(x))' + q(x)y_1(x) + \int_a^b G(x, t)y_1(t)dt = \lambda_1y_1(x), \quad (16) \]
and
\[ -(p(x)y_2'(x))' + q(x)y_2(x) + \int_a^b G(x,t)y_2(t)dt = \lambda_2 y_2(x). \quad (17) \]

From (16) and (17), we have
\[ (\lambda_1 - \lambda_2)y_1(x)y_2(x) = -(p(x)y_1'(x))'y_2(x) + (p(x)y_2'(x))'y_1(x) \]
\[ + y_2(x) \int_a^b G(x,t)y_1(t)dt - y_1(x) \int_a^b G(x,t)y_2(t)dt. \quad (18) \]
Integrating (18) on \([a, b]\), and using integration by parts and the boundary conditions, we have
\[ (\lambda_1 - \lambda_2) \int_a^b y_1(x)y_2(x)dx = \int_a^b (p(x)y_1'(x))'y_2(x)dx - \int_a^b (p(x)y_2'(x))'y_1(x)dx \]
\[ + \int_a^b y_2(x) \int_a^b g(x)g(t)y_1(t)dt dx - \int_a^b y_1(x) \int_a^b g(x)g(t)y_2(t)dt dx = 0. \quad (19) \]

Since \(\lambda_1 \neq \lambda_2\), (18) implies that \(\int_a^b y_1(x)y_2(x)dx = 0\).

This proves the Theorem.

To discuss the completeness of the eigenfunctions, we have to show that the eigenvalue sequence \(\lambda_n\) is unbounded.

**Lemma 3.2** Let \(\lambda_n(n = 1, 2, \cdots)\) be the \(n\)th eigenvalue of (1)-(2). We then have
\[ \lim_{n \to \infty} \lambda_n = \infty. \]

**Proof** Suppose \(y_n\) are the corresponding eigenfunctions of (1)-(2) of the eigenvalues \(\lambda_n(n = 1, 2, \cdots)\). From the definition of the \(\Omega_n(n = 1, 2, \cdots)\), we get that \(\{y_n\}\) is an orthogonal system in \(L^2[a, b]\). Now, we let \((y_i, y_j)_L = \delta_{ij}, i, j = 1, 2, \cdots\).

Suppose not, from Theorem 2.7, we have \(\{\lambda_n\}\) is a monotone increasing sequence. As
\[ \lambda_n \to \infty(n \to \infty), \]
then there exists \(\Lambda > 0\) such that
\[ \lim_{n \to \infty} \lambda_n = \Lambda, \quad \text{and} \quad \lambda_n \leq \Lambda, n = 1, 2, \cdots. \]

Using the definitions of \(\lambda_n\) and \(y_n\), we obtain that \(||y_n||_H = \sqrt{\lambda_n} \leq \sqrt{\Lambda}, n = 1, 2, \cdots\) i.e. \(\{y_n\}\) is a bounded sequence in \(H^1_{p,q}[a, b]\). Using Lemma 2.3, we have \(\{y_n\}\) is a compact sequence in \(C[a, b]\), and this implies \(\{y_n\}\) is a compact
sequence in $L^2[a, b]$. Hence, there is a subsequence $\{y_{n_j}\} \subset \{y_n\}$, which is convergent in $L^2[a, b]$. i.e.

$$||y_{n_i} - y_{n_j}||_L^2 \to 0 (i, j \to \infty).$$

On the other hand, we have

$$||y_{n_i} - y_{n_j}||_L^2 = ||y_{n_i}||_L^2 + ||y_{n_j}||_L^2 - 2(y_{n_i}, y_{n_j})_L = 2, \ \forall i, j = 1, 2, \cdots, i \neq j.$$

This gives the contradictions.

**Theorem 3.3** Let $y_n(x)$ be the eigenfunctions of (1)-(2) corresponding to the eigenvalues $\lambda_n(n = 1, 2, \cdots)$. Then, $\{y_n\}$ is a complete orthogonal system of eigenfunction in $L^2[a, b]$.

**Proof** The orthogonality is obvious. We only show the completeness. Since $L^2[a, b]$ is a Hilbert space, we only prove that the sequence $\{y_n\}$ is complete in $L^2[a, b]$.

Firstly, we prove that $y \in L^2 \cap H^{0,1}_{p,q,g}[a, b]$ with $(y, y_n)_L = 0, n = 1, 2, \cdots$, then $y = 0$.

Suppose not. Since $y$ is orthogonal to $y_n(n = 1, 2, \cdots)$, and $y_n \in \Omega_n, n = 1, 2, \cdots$, we obtain

$$\frac{(y, y)_H}{(y, y)_L} \geq \lambda_n, n = 1, 2, \cdots.$$  

From Theorem 3.2, we get $\lambda_n \to +\infty (n \to \infty)$. This gives the contradictions. Hence $y = 0$.

Secondly, since $H^{0,1}_{p,q,g}[a, b]$ is dense in $L^2[a, b]$, then, for $y \in L^2$ with $(y, y_n)_L = 0, n = 1, 2, \cdots$, we obtain $y = 0$. Therefore we conclude that $\{y_n\}$ is complete.

This finishes the proof.

**References**


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