Tripled Fixed Points Theorems for \((\phi, \psi)\)-Contractive Operators on Partially Ordered Metric Spaces without Mixed Monotone

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Abstract

In this paper, we prove tripled fixed point theorem for nonlinear \((\phi, \psi)\)-contractive on partially ordered complete metric spaces without mixed monotone. We also prove the uniqueness of tripled fixed point of such mappings. Further, we extend the recent results of the tripled fixed point theorem of the given mapping with mixed monotone in partially ordered metric spaces.

Keywords: tripled fixed point, partially ordered metric spaces, mixed monotone

1 Introduction

The fixed point problems of contractive mappings in partially ordered metric spaces has been considered recently by Ran and Reurings [1], Bhaskar and Lakshmikantham [2], Nieto and Lopez [3][4], Agarwal et al. [5], Berinde and Borcut [6] and P. Charoensawan [7].

Later in 2006, Bhaskar and Lakshmikantham [2], introduced the concept of a coupled fixed point and studied existence and uniqueness theorems in partially ordered metric spaces. They also applied their results to problems of the
existence of solution for a periodic boundary value problem.

V. Berinde and M. Borcut in [6] introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces and obtained existence.

Recently, P. Charoensawan [7] studied the existence of tripled fixed point of nonlinear mappings with mixed monotone and prove the uniqueness under some comparable conditions.

Let \((X, \leq)\) be a partially ordered set and suppose there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. On the product space \(X \times X \times X\), consider the following partial order :for \((x, y, z), (u, v, w) \in X \times X \times X\),

\[
(u, v, w) \leq (x, y, z) \iff x \geq u, y \leq v, z \geq w.
\]

**Definition 1.1** [6] Let \((X, \leq)\) be a partially ordered set and \(F : X \times X \times X \to X\). We say \(F\) has the mixed monotone property if for any \(x, y, z \in X\)

- \(x_1, x_2 \in X, x_1 \leq x_2 \) implies \(F(x_1, y, z) \leq F(x_2, y, z)\),
- \(y_1, y_2 \in X, y_1 \leq y_2 \) implies \(F(x, y_1, z) \geq F(x, y_2, z)\),
- and \(z_1, z_2 \in X, z_1 \leq z_2 \) implies \(F(x, y, z_1) \leq F(x, y, z_2)\).

**Definition 1.2** [6] An element \((x, y, z) \in X \times X \times X\) is called a tripled fixed point of a mapping \(F : X \times X \times X \to X\) if \(F(x, y, z) = x, F(y, x, y) = y\) and \(F(z, y, x) = z\).

Let \(\Phi\) denote the set of all functions \(\varphi : [0, \infty) \to [0, \infty)\) satisfying

- \((i_\varphi)\) \(\varphi\) is continuous and non-decreasing,
- \((ii_\varphi)\) \(\varphi(t) = 0\) if and only if \(t = 0\) and,
- \((iii_\varphi)\) \(\varphi(t + s) \leq \varphi(t) + \varphi(s)\) for all \(t, s \in [0, \infty)\)

and \(\Psi\) denote the set of all functions \(\psi : [0, \infty) \to [0, \infty)\) which satisfy

- \((i_\psi)\) \(\lim_{t \to r^+} \psi(t) > 0\) for all \(r > 0\) and \((ii_\psi)\) \(\lim_{t \to 0^+} \psi(t) = 0\).

**Theorem 1.3** [7] Let \((X, \leq)\) be a partially ordered set and suppose there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(F : X \times X \to X\) be a mixed monotone mapping for which there exist \(\varphi \in \Phi\) and \(\psi \in \Psi\) such that for all \(x, y, z, u, v, w \in X\) with \(x \geq u, y \leq v\) and \(z \geq w\)

\[
\varphi\left(\frac{d(F(x, y, z), F(u, v, w) + d(F(y, x, y), F(v, u, v) + d(F(z, y, x), F(w, v, u))}{3}\right) \\
\leq \varphi\left(\frac{d(x, u) + d(y, v) + d(z, w)}{3}\right) - \psi\left(\frac{d(x, u) + d(y, v) + d(z, w)}{3}\right). \quad (1)
\]
suppose either

(a) $F$ is continuous or
(b) $X$ has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n$.
(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n$.

If there exist $x_0, y_0, z_0 \in X$ such that

\[ x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad z_0 \leq F(z_0, y_0, x_0), \]

or

\[ x_0 \geq F(x_0, y_0, z_0), \quad y_0 \leq F(y_0, x_0, y_0) \quad \text{and} \quad z_0 \geq F(z_0, y_0, x_0), \]

Then there exist $x, y, z \in X$ such that

\[ x = F(x, y, z), \quad y = F(y, x, y) \quad \text{and} \quad z = F(z, y, x). \]

2 Main Results

We give the notion of an $F$-invariant set is useful for main results.

Definition 2.1 Let $(X, d)$ be a metric space and $F : X \times X \to X$ be a given mapping. Let $M$ be a nonempty subset of $X^6$. We say that $M$ is an $F$-invariant subset of $X^6$ if and only if, for all $x, y, z, u, v, w \in X$, $(x, y, z, u, v, w) \in M$ \Rightarrow $(F(x, y, z), F(y, x, y), F(z, y, x), F(u, v, w), F(v, u, v), F(w, v, u)) \in M$.

Definition 2.2 Let $(X, d)$ be a metric space and $M$ be a subset of $X^6$. We say that $M$ satisfies the transitive property if and only if, for all $x, y, z, u, v, w, a, b, c \in X$, $(x, y, z, u, v, w) \in M$ and $(u, v, w, a, b, c) \in M$ \Rightarrow $(x, y, z, a, b, c) \in M$.

Remark The set $M = X^6$ is trivially $F$-invariant, which satisfies the transitive property.

Example 2.3 Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \times X \to X$ be a mapping. Define a subset $M \subseteq X^6$ by $M = \{(a, b, c, d, e, f) \in X^6 : a \geq d, b \leq e, c \geq f \}$ Then $M$ is an $F$-invariant subset of $X^6$, which satisfies the transitive property.
Our first main result is the following:

**Theorem 2.4** Let \((X, \leq)\) be a partially ordered set and suppose there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(F : X \times X \to X\) be mapping for which there exist \(\varphi \in \Phi\) and \(\psi \in \Psi\) such that for all \(x, y, z, u, v, w \in X\) with \((x, y, z, u, v, w) \in M\).

\[
\varphi\left(\frac{d(F(x, y, z), F(u, v, w) + d(F(y, x, y), F(v, u, v) + d(F(z, y, x), F(w, v, u))}{3}\right) \\
\leq \varphi\left(\frac{d(x, u) + d(y, v) + d(z, w)}{3}\right) - \psi\left(\frac{d(x, u) + d(y, v) + d(z, w)}{3}\right).
\]

suppose either

(a) \(F\) is continuous or

(b) for any three sequences \(\{x_n\}, \{y_n\}, \{z_n\}\) with \((x_{n+1}, y_{n+1}, z_{n+1}, x_n, y_n, z_n) \in M\), \(\{x_n\} \to x, \{y_n\} \to y\) and \(\{z_n\} \to z\) for all \(n \geq 1\) implies \((x, y, z, x_n, y_n, z_n) \in M\) for all \(n \geq 1\).

If there exist \((x_0, y_0, z_0) \in X \times X \times X\) such that
\((F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)), x_0, y_0, z_0) \in M\) and \(M\) is an \(F\)-invariant set which satisfies the transitive property. Then there exist \(x, y, z \in X\) such that

\[x = F(x, y, z),\ y = F(y, x, y)\ and\ z = F(z, y, x)\]

**Proof.**

Let \((x_0, y_0, z_0) \in X \times X \times X\). Since \(F(X \times X \times X) \subseteq X\), we can choose \(x_1, y_1, z_1 \in X\) such that

\[x_1 = F(x_0, y_0, z_0),\ y_1 = F(y_0, x_0, y_0)\ and\ z_1 = F(z_0, y_0, x_0)\]

Again from \(F(X \times X \times X) \subseteq X\) we can choose \(x_2, y_2, z_2 \in X\) such that

\[x_2 = F(x_1, y_1, z_1),\ y_2 = F(y_1, x_1, y_1)\ and\ z_2 = F(z_1, y_1, x_1)\]

Continuing this process we can construct sequences \(\{x_n\}, \{y_n\}\) and \(\{z_n\}\) in \(X\) such that

\[x_n = F(x_{n-1}, y_{n-1}, z_{n-1}),\ y_n = F(y_{n-1}, x_{n-1}, y_{n-1})\ and\ z_n = F(z_{n-1}, y_{n-1}, x_{n-1})\]

for all \(n \geq 1\).

If there exists \(k \in \mathbb{N}\) such that \(x_k = x_{k-1},\ y_k = y_{k-1}\) and \(z_k = z_{k-1}\), then
\(x_k = x_{k-1} = F(x_{k-1}, y_{k-1}, z_{k-1}),\ y_k = y_{k-1} = F(y_{k-1}, x_{k-1}, y_{k-1})\) and \(z_k = z_{k-1} = F(z_{k-1}, y_{k-1}, x_{k-1})\).
Thus, \((x_{k-1}, y_{k-1}, z_{k-1})\) is a tripled fixed point of \(F\). This is finishes the proof. Therefore, we may assume that \(x_k \neq x_{k-1}\) or \(y_k \neq y_{k-1}\) or \(z_k \neq z_{k-1}\) for all \(n \geq 1\).

Since \((F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0), x_0, y_0, z_0) = (x_1, y_1, z_1, x_0, y_0, z_0) \in M\) and \(M\) is an \(F\)-invariant set, we have

\[(F(x_1, y_1, z_1), F(y_1, x_1, y_1), F(z_1, y_1, x_1), F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)) = (x_2, y_2, z_2, x_1, y_1, z_1) \in M.\]

By repeating this argument, we get

\[(F(x_{n-1}, y_{n-1}, z_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), F(z_{n-1}, y_{n-1}, x_{n-1}), F(x_n, y_n, z_n), F(y_n, x_n, y_n), F(z_n, y_n, x_n)) = (x_n, y_n, z_n, x_{n-1}, y_{n-1}, z_{n-1}) \in M\] for all \(n \geq 1\).

Consider now the sequence of nonnegative real number \(\{\delta_n\}_{n=1}^{\infty}\) given by

\[
\delta_{n+1} = \frac{d(x_{n+1}, x_n) + d(y_{n+1}, y_n) + d(z_{n+1}, z_n)}{3}, \quad n \geq 0. \tag{4}
\]

Since \((x_n, y_n, z_n, x_{n-1}, y_{n-1}, z_{n-1}) \in M\), we have

\[
\frac{d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1}))) + d(F(y_n, x_n, y_n), F(y_{n-1}, x_{n-1}, y_{n-1}))) + d(F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1}))}{3} = \delta_{n+1}.
\]
While the right hand side of (2) will be equal to

\[
\varphi\left(\frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1})}{3}\right) - \psi\left(\frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1})}{3}\right)
\]

\[
= \varphi(\delta_n) - \psi(\delta_n).\]
Therefore, the sequence \(\{\delta_n\}_{n=1}^{\infty}\) satisfies

\[
\varphi(\delta_{n+1}) \leq \varphi(\delta_n) - \psi(\delta_n) \leq \varphi(\delta_n), \tag{5}
\]
for all \(n \geq 0\).

From (5) and \((i)_{\varphi}\) it follows that the sequence \(\{\delta_n\}_{n=1}^{\infty}\) is non-increasing. Therefore, there is some \(\delta \geq 0\) such that

\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \frac{d(x_{n+1}, x_n) + d(y_{n+1}, y_n) + d(z_{n+1}, z_n)}{3} = \delta. \tag{6}
\]

We shall prove that \(\delta = 0\). Suppose, to the contrary, that \(\delta > 0\). Then taking the limit as \(n \to \infty\) (equivalently, \(\delta_n \to \delta\)) of both side of (5) and by property \((i_{\varphi})\) and \((i_{\psi})\), we have

\[
\varphi(\delta) = \lim_{n \to \infty} \varphi(\delta_n) \leq \lim_{n \to \infty} [\varphi(\delta_{n-1}) - \psi(\delta_{n-1})] = \varphi(\delta) - \lim_{\delta_{n-1} \to \delta} \psi(\delta_{n-1}) < \varphi(\delta)
\]
which is a contradiction. Thus \(\delta = 0\), that is,

\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \frac{d(x_{n+1}, x_n) + d(y_{n+1}, y_n) + d(z_{n+1}, z_n)}{3} = 0. \tag{7}
\]
We now prove that \( \{a_n\} \) is a Cauchy sequence in \((X^3, d_3)\), that is, \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) are Cauchy sequences. Suppose, to the contrary, that is at least of \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) is not Cauchy sequence. Then there exists an \( \varepsilon > 0 \) for which we can find subsequences \( \{x_{n(k)}\} \) and \( \{x_{m(k)}\} \) of \( \{x_n\} \), \( \{y_{n(k)}\} \) and \( \{y_{m(k)}\} \) of \( \{y_n\} \) and \( \{z_{n(k)}\} \) and \( \{z_{m(k)}\} \) of \( \{z_n\} \) with \( n(k) > m(k) \geq K \) such that

\[
\frac{1}{3} \left[ d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) \right] \geq \varepsilon. \tag{8}
\]

Further, corresponding to \( m(k) \), we can choose \( n(k) \) in such a way that is the smallest integer with \( n(k) > m(k) \geq K \) and satisfying (8). Then

\[
\frac{1}{3} \left[ d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)}) + d(z_{n(k)-1}, z_{m(k)}) \right] < \varepsilon. \tag{9}
\]

Using (8) and (9) and the triangle inequality, we have

\[
\varepsilon \leq r_k := \frac{1}{3} \left[ d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) \right]
\]

\[
\leq \frac{1}{3} \left[ d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) + d(z_{n(k)}, z_{n(k)-1}) + d(z_{n(k)-1}, z_{m(k)}) \right]
\]

\[
\leq \frac{1}{3} \left[ d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1}) + d(z_{n(k)}, z_{n(k)-1}) + \varepsilon \right].
\]

Letting \( k \to \infty \) and using (7), we get

\[
\varepsilon \leq \lim_{k \to \infty} r_k \leq \lim_{k \to \infty} \left[ \frac{d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1}) + d(z_{n(k)}, z_{n(k)-1})}{3} + \varepsilon \right] = \varepsilon,
\]

that is

\[
\lim_{k \to \infty} r_k = \lim_{k \to \infty} \left[ \frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)})}{3} \right] = \varepsilon. \tag{10}
\]

By the triangle inequality, we have

\[
d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)}),
\]

similarly

\[
d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)}),
\]

and

\[
d(z_{n(k)}, z_{m(k)}) \leq d(z_{n(k)}, z_{n(k)+1}) + d(z_{n(k)+1}, z_{m(k)+1}) + d(z_{m(k)+1}, z_{m(k)}).
\]
This shows that

\[
\begin{align*}
    r_k & = \frac{d(x_n(k), x_m(k)) + d(y_n(k), y_m(k)) + d(z_n(k), z_m(k))}{3} \\
    & \leq \frac{d(x_n(k), x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_m(k))}{3} \\
    & \quad + \frac{d(y_n(k), y_{n(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_m(k))}{3} \\
    & \quad + \frac{d(z_n(k), z_{n(k)+1}) + d(z_{n(k)+1}, z_{m(k)+1}) + d(z_{m(k)+1}, z_m(k))}{3} \\
    & = \delta_n(k) + \delta_m(k) \\
    & \quad + \frac{d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}) + d(z_{n(k)+1}, z_{m(k)+1})}{3}.
\end{align*}
\] (11)

Since \( n(k) > m(k) \) and \( M \) satisfies the transitive property, from

\[
\begin{align*}
    (x_n(k), y_n(k), z_n(k), x_{n(k)-1}, y_{n(k)-1}, z_{n(k)-1}) \in M \quad \text{and} \\
    (x_{m(k)+1}, y_{m(k)+1}, z_{m(k)+1}, x_m(k), y_m(k), z_m(k)) \in M.
\end{align*}
\]

We have \((x_n(k), y_n(k), z_n(k), x_m(k), y_m(k), z_m(k)) \in M\). By (2) we get

\[
\begin{align*}
    \varphi \left( \frac{d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}) + d(z_{n(k)+1}, z_{m(k)+1})}{3} \right) \\
    & = \varphi \left( \frac{d(F(x_n(k), y_n(k), z_n(k)), F(x_m(k), y_m(k), z_m(k)))}{3} \right) \\
    & \quad + \frac{d(F(y_n(k), x_n(k), y_n(k)), F(y_m(k), x_m(k), y_m(k)))}{3} \\
    & \quad + \frac{d(F(z_n(k), y_n(k), x_n(k)), F(z_m(k), y_m(k), x_m(k)))}{3} \\
    & \leq \varphi \left( \frac{d(x_n(k), x_m(k)) + d(y_n(k), y_m(k)) + d(z_n(k), z_m(k))}{3} \right) \\
    & \quad - \psi \left( \frac{d(x_n(k), x_m(k)) + d(y_n(k), y_m(k)) + d(z_n(k), z_m(k))}{3} \right) \\
    & = \varphi(r_k) - \psi(r_k). \quad \text{(12)}
\end{align*}
\]

On the other hand, by (ref11) and using property (iii\(\varphi\)), we get

\[
\begin{align*}
    \varphi(r_k) & \leq \varphi(\delta_n(k) + \delta_m(k)) \\
    & \quad + \varphi \left( \frac{d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}) + d(z_{n(k)+1}, z_{m(k)+1})}{3} \right). \quad \text{(13)}
\end{align*}
\]

By (12) and (13), we have

\[
\begin{align*}
    \varphi(r_k) & \leq \varphi(\delta_n(k) + \delta_m(k)) + \varphi(r_k) - \psi(r_k) \quad \text{(14)}
\end{align*}
\]

**Tripled fixed points theorems**
Letting $k \to \infty$ in (14) and using (7) and (10) and property of $\varphi$ and $\psi$, we have

$$\varphi(\varepsilon) = \lim_{k \to \infty} \varphi(r_k) = \varphi(\lim_{k \to \infty} r_k) \leq \lim_{k \to \infty} [\varphi(\delta_n(k) + \delta_m(k)) + \varphi(r_k) - \psi(r_k)] = \varphi(\lim_{k \to \infty} (\delta_n(k) + \delta_m(k))) + \varphi(\lim_{k \to \infty} r_k) - \lim_{k \to \infty} \psi(r_k) = \varphi(0) + \varphi(\varepsilon) - \lim_{k \to \infty} \psi(r_k) < \varphi(\varepsilon),$$

which is a contradiction. This shows that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are Cauchy sequences. Since $X$ is a complete metric space, there exist $x, y \in X$ such that

$$\lim_{n \to \infty} x_n = x, \lim_{n \to \infty} y_n = y \text{ and } \lim_{n \to \infty} z_n = z.$$

Now suppose that assumption (a) holds. Then

$$x = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} F(x_n, y_n, z_n) = F(x, y, z)$$

$$y = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} F(y_n, x_n, y_n) = F(y, x, y)$$

$$z = \lim_{n \to \infty} z_{n+1} = \lim_{n \to \infty} F(z_n, y_n, x_n) = F(z, y, x).$$

Suppose assumption (b) holds. We obtain that a sequence $\{x_n\} \to x$, $\{y_n\} \to y$ and $\{z_n\} \to z$ by the assumption, we have $(x, y, z, x_n, y_n, z_n) \in M$ for all $n \geq 1$.

By $\varphi$ is non-decreasing and (2), therefore

$$\varphi \left( \frac{d(x, F(x, y, z)) - d(x, x_{n+1}) + d(y, F(y, x, y)) - d(y, y_{n+1})}{3} + \frac{d(z, F(z, y, x)) - d(z, z_{n+1})}{3} \right)$$

$$\leq \varphi \left( \frac{d(F(x_n, y_n, z_n), F(x, y, z)) + d(F(y_n, x_n, y_n), F(y, x, y))}{3} + \frac{d(F(z_n, y_n, x_n), F(z, y, x))}{3} \right)$$

$$\leq \varphi \left( \frac{d(x_n, x) + d(y_n, y) + d(z_n, z)}{3} - \psi \left( \frac{d(x_n, x) + d(y_n, y) + d(z_n, z)}{3} \right) \right)$$

$$\leq \varphi \left( \frac{d(x_n, x) + d(y_n, y) + d(z_n, z)}{3} \right).$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain

$$\varphi \left( \frac{d(x, F(x, y, z)) + d(y, F(y, x, y)) + d(z, F(z, y, x))}{3} \right) \leq \varphi(0) = 0$$
which shows, by \((i_\varphi)\) and \((iii_\varphi)\), that
\[
x = F(x, y, z), y = F(y, x, y) \quad \text{and} \quad z = F(z, y, x).
\]

\[\blacksquare\]

**Theorem 2.5** In addition to the hypotheses of theorem 2.1, suppose that for every \((x, y, z), (x^*, y^*, z^*) \in X \times X \times X\) there exists a \((u, v, w) \in X \times X \times X\) such that \((x, y, z, u, v, w) \in M\) and \((x^*, y^*, z^*, u, v, w) \in M\). Then \(F\) have a unique tripled fixed point.

**Proof.** From Theorem 2.1, the set of tripled fixed points of \(F\) is nonempty. Assume that \((x, y, z)\) and \((x^*, y^*, z^*) \in X \times X \times X\) are two tripled fixed points of \(F\). We shall prove that \(x = x^*, y = y^*\) and \(z = z^*\).

Since \((x, y, z)\) and \((x^*, y^*, z^*) \in X \times X \times X\) there exists \((u, v, w) \in X \times X \times X\) such that \((u, v, w)\) is comparable to \((x, y, z)\) and \((x^*, y^*, z^*)\). We define the sequence \(\{u_n\}, \{v_n\}\) and \(\{w_n\}\) as follows:

\[
u_0 = u, v_0 = v, w_0 = w, u_{n+1} = F(u_n, v_n, w_n), v_{n+1} = F(v_n, u_n, v_n) \quad \text{and} \quad w_{n+1} = F(w_n, v_n, u_n)\]

for all \(n \geq 0\).

Further, set \(x_0 = x, y_0 = y, z_0 = z, x_0^* = x^*, y_0^* = y^*, z_0^* = z^*\) and, on the same way, define the sequences \(\{x_n\}, \{y_n\}, \{z_n\}, \{x_n^*\}, \{y_n^*\}\) and \(\{z_n^*\}\). That is, as above,

\[
\text{for all } n \geq 0, x_{n+1} = F(x_n, y_n, z_n), y_{n+1} = F(y_n, x_n, y_n), z_{n+1} = F(z_n, y_n, x_n) \quad \text{and} \quad x_{n+1}^* = F(x_n^*, y_n^*, z_n^*), y_{n+1}^* = F(y_n^*, x_n^*, y_n^*), z_{n+1}^* = F(z_n^*, y_n^*, x_n^*).
\]

Since \(M\) is \(F\)-invariant and \((x, y, z, u, v, w) = (x, y, z, u_0, v_0, w_0) \in M\), we have \((F(x, y, z), F(y, x, y), F(z, y, x), F(u_0, v_0, w_0), F(v_0, u_0, v_0), F(w_0, v_0, u_0)) = (x, y, z, u_1, v_1, w_1) \in M\). It is easy to show that \((x, y, z, u_n, v_n, w_n) \in M\) Therefore, by (2.1),

\[
\varphi\left(\frac{d(x, u_{n+1}) + d(y, v_{n+1}) + d(z, w_{n+1})}{3}\right) = \varphi\left(\frac{d(F(x, y, z), F(u_n, v_n, w_n) + d(F(y, x, y), F(v_n, u_n, v_n)}{3} + \right.
\]

\[
\frac{d(F(z, y, x), F(w_n, v_n, u_n))}{3} \leq \varphi\left(\frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3}\right) - \psi\left(\frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3}\right).
\]

Using the property of \(\psi\), we get

\[
\varphi\left(\frac{d(x, u_{n+1}) + d(y, v_{n+1}) + d(z, w_{n+1})}{3}\right) \leq \varphi\left(\frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3}\right).
\]

\[
(15)
\]
By \( \varphi \) is non-decreasing, therefore

\[
\frac{d(x, u_{n+1}) + d(y, v_{n+1}) + d(z, w_{n+1})}{3} \leq \frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3}.
\]

Denote

\[
\delta_n = \frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3}, \quad n \geq 0.
\]

So \( \{\delta_n\} \) is non-decreasing. Hence, there exists \( \alpha \geq 0 \) such that

\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \left[ \frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3} \right] = \alpha. \quad (16)
\]

We shall show that \( \alpha = 0 \). Suppose, to the contrary, that \( \alpha > 0 \). Taking the limit as \( n \to \infty \) in (15). By (16), we have

\[
\varphi(\alpha) = \lim_{n \to \infty} \varphi\left( \frac{d(x, u_{n+1}) + d(y, v_{n+1}) + d(z, w_{n+1})}{3} \right)
\leq \lim_{n \to \infty} \varphi\left( \frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3} \right)
- \lim_{n \to \infty} \psi\left( \frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3} \right)
= \varphi(\alpha) - \lim_{n \to \infty} \psi\left( \frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3} \right).
\]

By property of \( \psi \), we get \( \varphi(\alpha) < \varphi(\alpha) \). Which is a contradiction. Thus \( \alpha = 0 \), that is,

\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \left[ \frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3} \right] = 0.
\]

which implies

\[
\lim_{n \to \infty} d(x, u_n) = \lim_{n \to \infty} d(y, v_n) = \lim_{n \to \infty} d(z, w_n) = 0
\]

Similarly, we obtain

\[
\lim_{n \to \infty} d(x^*, u_n) = \lim_{n \to \infty} d(y^*, v_n) = \lim_{n \to \infty} d(z^*, w_n) = 0,
\]

and hence \( x = x^*, y = y^* \) and \( z = z^* \).

**Example 2.6** Let \( X = \mathbb{R}, d(x, y) = |x - y| \) and \( F : X \times X \times X \to X \) be defined by

\[
F(x, y, z) = \frac{x + y + z}{18}, \quad (x, y, z) \in X^3.
\]

The mapping \( F \) does not satisfy the mixed monotone property. It is easy to check that \( F \) satisfies (2) with \( M = X^6 \), \( \varphi(t) = \frac{t}{2} \) and \( \psi(t) = \frac{t}{3} \) and \( (0,0,0) \) is the unique tripled fixed point of \( F \).
Next, we give a simple application of our results to tripled fixed point theorems in partially ordered metric spaces.

**Corollary 2.7** Let \((X, \leq)\) be a partially ordered set and suppose there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(F : X \times X \to X\) be a mixed monotone mapping for which there exist \(\phi \in \Phi\) and \(\psi \in \Psi\) such that for all \(x, y, z, u, v, w \in X\) with \(x \geq u, y \leq v\) and \(z \geq w\)

\[
\phi\left(\frac{d(F(x, y, z), F(u, v, w) + d(F(y, x, y), F(v, u, v) + d(F(z, y, x), F(w, v, u))}{3}\right) \\
\leq \phi\left(\frac{d(x, u) + d(y, v) + d(z, w)}{3}\right) - \psi\left(\frac{d(x, u) + d(y, v) + d(z, w)}{3}\right).
\]

(17)

suppose either

(a) \(F\) is continuous or

(b) \(X\) has the following property:

(i) if a non-decreasing sequence \(\{x_n\} \to x\), then \(x_n \leq x\) for all \(n\).

(ii) if a non-increasing sequence \(\{y_n\} \to y\), then \(y \leq y_n\) for all \(n\).

If there exist \(x_0, y_0, z_0 \in X\) such that

\(x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0)\) and \(z_0 \leq F(z_0, y_0, x_0),\)

or

\(x_0 \geq F(x_0, y_0, z_0), y_0 \leq F(y_0, x_0, y_0)\) and \(z_0 \geq F(z_0, y_0, x_0),\)

Then there exist \(x, y, z \in X\) such that

\(x = F(x, y, z), y = F(y, x, y)\) and \(z = F(z, y, x).\)

**Proof.** Define a subset \(M \subseteq X^6\) by \(M = \{(a, b, c, d, e, f) \in X^6 : a \geq d, b \leq e, c \geq f\}\),

then \(M\) is an \(F\)-invariant set which satisfies the transitive property. By (17),

we have

\[
\phi\left(\frac{d(F(x, y, z), F(u, v, w) + d(F(y, x, y), F(v, u, v) + d(F(z, y, x), F(w, v, u))}{3}\right) \\
\leq \phi\left(\frac{d(x, u) + d(y, v) + d(z, w)}{3}\right) - \psi\left(\frac{d(x, u) + d(y, v) + d(z, w)}{3}\right).
\]

For all \(x, y, z, u, v, w \in X\) with \((x, y, z, u, v, w) \in M\).

Since \(x_0, y_0, z_0 \in X\) such that

\(x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0)\) and \(z_0 \leq F(z_0, y_0, x_0).\)
We get \((F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0), x_0, y_0, z_0) \in M)\).

If \(F\) is continuous, then we finish the proof.

Assume \(F\) is not continuous. For any three sequences \(\{x_n\}, \{x_n\}, \{z_n\}\) such that \(\{x_n\}\) is non-decreasing sequence \(x_n \to x\), \(\{y_n\}\) is non-increasing sequence \(y_n \to y\) and \(\{z_n\}\) is non-decreasing sequence \(z_n \to z\). We have

\[
\begin{align*}
x_1 &\leq x_2 \leq \ldots \leq x_n \leq \ldots \leq x, \\
y_1 &\geq y_2 \geq \ldots \geq y_n \geq \ldots \geq y, \\
z_1 &\leq z_2 \leq \ldots \leq z_n \leq \ldots \leq z.
\end{align*}
\]

for all \(n \geq 1\). Therefore, we have \((x, y, z, x_n, y_n, z_n) \in M\) for all \(n \geq 1\), and so the assumption of Theorem 2.1 (b) holds, thus \(F\) has a tripled fixed point. \(\blacksquare\)

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**References**


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