A Single Server M/G/1 Queue with Preemptive Repeat Priority Service with Server Vacation

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Abstract

This paper pertains to the study of a Stochastic queueing model with preemptive repeat priority service discipline which plays a prominent role in real life situation like communication network where priority is given important when urgent message is to be communicated, man power planning where the recruitment process following preemptive repeat priority service discipline, VIP’s vehicle during peak hours etc. Here an M/G/1 queue with optional server vacation based on Bernoulli schedule with single server providing two types of service – High priority service (Type I) with probability  \( p_1 \) and low priority service (Type II) with probability  \( p_2 \) with service time following general distribution. Further it is assume that the low priority service will be interrupted when a high priority customer arrives in the system. The time dependent probability generating function have been obtained in terms of their Laplace transform and the corresponding steady state results are obtained explicitly for this model. Also some performance measures such as mean queue length and mean waiting time are computed for high priority and low priority queue. The validity of this model is highlighted by means of a hypothetical situation.
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Introduction

Priority queueing models play an vital role in the day to day life and it has been studied and investigated by many researchers. One such model is developed here which employs preemptive repeat service discipline to serve customer based on high priority and low priority. That is M/G/1 queue with optional server vacation based on Bernoulli schedule with single server providing two types of service – High priority service (Type I) with probability $p_1$ and low priority service (Type II) with probability $p_2$ with service time following general distribution. Further it is assume that the low priority service will be interrupted when a high priority customer arrives in the system.

Again at the completion of each type of service server can take a single vacation with probability $\theta$ or may continue to stay in the system with probability $(1-\theta)$. Further the low priority unit can get his service repeated with probability $p$ until it is successful or may depart from the system with probability $q=(1-p)$ if the service happens to be successful as in the case of Stochastic queueing models subject to Bernoulli feedback which has many real life application.

The model under consideration is based on the following priority queueing model.

Priority queueing models with Poisson arrival and general service time distribution has been investigated by Miller [7], Jaiswal [3] and Tackacs [8].

Gaver.D.P [2] dealt with waiting lines with interrupted service including priorities. Simon (1984) has analyzed an M/G/1 feedback queue with multiple customer types and pre-emptive and non-pre-emptive priority levels that may change after a service completion with customers had feedback a fixed number of times.

Choe.Y.Z.etal [1] considered a combined preemptive/non preemptive priority discipline in the analysis of the M/G/1 queue and considered preemptive rule that are preemptive-resume and preemptive-repeat-identical policies.

Liping wang [6] [2012] considered M/G/1 queue with vacation and priorities and introduced the concept of vacation, non-preemptive priority, preemptive-resume priority and shown how to derive the average remaining vacation time and average waiting time for M/G/1 queues with vacation and the average waiting time and the
average system time for M/G/1 queues with non-preemptive or preemptive-resume priority.

Asymptotics for M/G/1 low priority waiting-time tail probabilities were considered by Joseph abate and Ward Whitt in the year [4].

Weichang [9] considered preemptive priority queues where it is assume that the customers of different priorities arrive at a counter in accordance with a Poisson process. The customers served by a single server in order of priority and for each priority in order of arrival. Preemptive discipline is assumed. Three service policies are considered

(i) Preemptive-resume
   (ii) Preemptive-repeat-identical
   and (iii)Preemptive-repeat-different

One such model is developed here which employs preemptive repeat service discipline to serve customers based on high priority and low priority. Here the transient state behavior of the model is studied using the differential difference equations and probability generating function technique. Customers arrive in a Poisson fashion and they are segregated as high priority and low priority customers. They wait in two separate queues (say) $q_1$ and $q_2$ respectively to get service from the server.

Mathematical description of the model:

In this paper the following are assumed to describe the queueing model
1. The arrival process of the high priority and low priority customers follows Poisson distribution.
2. Service time follows general distribution.
3. Queue discipline follows pre-emptive repeat priority rule.
4. A high priority customer is served in the order of arrival in the queue $q_1$.
5. A low priority customer is admitted into the service in the order of arrival in the queue $q_2$.
6. Further we assume when a low priority customer is in service and if a high priority customer arrives in $q_1$ then the service to the low priority customer is interrupted and high priority customer is served.
7. After the completion of service to high priority customer, the low priority customer whose service is interrupted can get his service repeated with probability $p$ and he can depart from the system with probability $q = (1-p)$ if the service happened to be successful provided there is no customer for high priority service.
8. Customers arrive at the system one by one according to a Poisson stream with arrival rate $\lambda (>0)$. 


9. The server provides two types of service, type I and type II with the service times having general distribution. Let $B_1(v)$ and $b_1(v)$ respectively be the distribution and the density function of the type I service.

10. The service time of type II is also assumed to be general with the distribution function $B_2(v)$ and the density function $b_2(v)$.

11. It is assumed that the server provides two types of service type I with probability $p_1$ and Type II with probability $p_2$ for high and low priority customers respectively.

12. Further $\mu_i(x)dx$ is the probability of completion of the $i$th type service given that elapsed time is $x$, so that

$$\mu_i(x) = \frac{b_i(x)}{1-B_i(x)} , \quad i = 1, 2$$

and therefore

$$b_i(v) = \mu_i(v) e^{-\int_0^v \mu_i(x)dx} , \quad i = 1, 2$$

13. After the completion of each type of service, the server may take a vacation with probability $\theta$ or may continue staying in the system with probability $(1-\theta)$.

14. The vacation periods are exponentially distributed with mean vacation time $\frac{1}{\beta}$.

15. On returning from vacation the server instantly starts servicing the customer. The inter arrival time, the service times of each type of service and the vacation times are independent of each other.

16. The various stochastic processes involved in the system are independent of each other.

**Equations governing the system:**

Define

$$P_{n_i}^{(1)}(x,t) = \text{Probability that at time } t, \text{ there are } n_i (\geq 0) \text{ high priority customers in the queue excluding one customer in the first type of service and the elapsed service time for this customer is } x.$$  

Consequently, $P_{n_i}^{(1)}(t) = \int_0^\infty P_{n_i}^{(1)}(x,t)dx$ denotes the probability that at time $t$ there are $n_i > 0$ customers in the queue excluding the one customer in the high priority service (Type I) irrespective of the value of $x$.

$$P_{n_2}^{(2)}(x,t) = \text{Probability that at time } t, \text{ there are } n_2 (\geq 0) \text{ low priority customers in the queue excluding one customer in the second type of service and the elapsed service time for this customer is } x.$$
Consequently, \( P^{(1)}_{n_1}(t) = \int_0^\infty P^{(1)}_{n_1}(x,t)dx \) denotes the probability that at time \( t \) there are \( n_1 > 0 \) customers in the queue excluding the one customer in the low priority service (Type II) irrespective of the value of \( x \).

\( V_{n_1}(t) \) = Probability that at time \( t \), there are \( n_1 \geq 0 \) customers in the high priority queue \( q_1 \) and the server is on vacation.

\( V_{n_2}(t) \) = Probability that at time \( t \), there are \( n_2 \geq 0 \) customers in the low priority queue \( q_2 \) and the server is on vacation.

\( Q(t) \) = Probability that at time \( t \), there is no customer in the queue and the server is idle but available in the system.

The model is then, governed by the following set of differential – difference equations for \( q_1 \) and \( q_2 \) respectively.

\[
\frac{\partial}{\partial x} P^{(1)}_{n_1}(x,t) + \frac{\partial}{\partial t} P^{(1)}_{n_1}(x,t) + \left[ \dot{\lambda} + \mu_1(x) \right] P^{(1)}_{n_1}(x,t) = \lambda P^{(1)}_{n_1-1}(x,t)
\]

(3)

\[
\frac{\partial}{\partial x} P^{(1)}_{0}(x,t) + \frac{\partial}{\partial t} P^{(1)}_{0}(x,t) + \left[ \dot{\lambda} + \mu_1(x) \right] P^{(1)}_{0}(x,t) = 0
\]

(4)

\[
\frac{d}{dt} V_{n_1}(t) = - (\dot{\lambda} + \beta) V_{n_1}(t) + V_{n_1-1}(t) \dot{\lambda} + \theta \int_0^\infty P^{(1)}_{n_1}(x,t) \mu_1(x)dx
\]

(5)

\[
\frac{d}{dt} V_{0}(t) = - (\dot{\lambda} + \beta) V_{0}(t) + \theta \int_0^\infty P^{(1)}_{0}(x,t) \mu_1(x)dx
\]

(6)

\[
\frac{\partial}{\partial x} P^{(2)}_{n_2}(x,t) + \frac{\partial}{\partial t} P^{(2)}_{n_2}(x,t) + \left[ \dot{\lambda} + \mu_2(x) \right] P^{(2)}_{n_2}(x,t) = \lambda P^{(2)}_{n_2-1}(x,t)
\]

(7)

\[
\frac{\partial}{\partial x} P^{(2)}_{0}(x,t) + \frac{\partial}{\partial t} P^{(2)}_{0}(x,t) + \left[ \dot{\lambda} + \mu_2(x) \right] P^{(2)}_{0}(x,t) = 0
\]

(8)

\[
\frac{d}{dt} V_{n_2}(t) = - (\dot{\lambda} + \beta) V_{n_2}(t) + V_{n_2-1}(t) \dot{\lambda} + \theta \int_0^\infty P^{(2)}_{n_2}(x,t) \mu_2(x)dx
\]

(9)

\[
\frac{d}{dt} V_{0}(t) = - (\dot{\lambda} + \beta) V_{0}(t) + \theta \int_0^\infty P^{(2)}_{0}(x,t) \mu_2(x)dx
\]

(10)

For \( q_1 \) and \( q_2 \)

\[
\frac{d}{dt} Q(t) = -\dot{\lambda} Q(t) + \beta V_{0}(t) + (1 - \theta) \int_0^\infty P^{(1)}_{0}(x,t) \mu_1(x)dx + (1 - \theta) q \int_0^\infty P^{(2)}_{0}(x,t) \mu_2(x)dx
\]

(11)

Boundary conditions
\begin{align*}
P^{(1)}_0(0,t) &= Q_i(t) \lambda p_1 + p_1 \beta V_1(t) + p_1(1-\theta) \int_0^\infty P^{(1)}_0(x,t) \mu_1(x) dx \\
P^{(1)}_{n_1}(0,t) &= p_1 \beta V_{n_1+1}(t) + p_1(1-\theta) \int_0^\infty P^{(1)}_{n_1}(x,t) \mu_1(x) dx \\
P^{(2)}_0(0,t) &= Q_2(t) \lambda p_2 + p_2 \beta V_1(t) + p_2(1-\theta) p_1 \int_0^\infty P^{(2)}_0(x,t) \mu_2(x) dx + p_2(1-\theta) q_1 \int_0^\infty P^{(2)}_{n_1}(x,t) \mu_2(x) dx \\
P^{(2)}_{n_2+1}(0,t) &= p_2 \beta V_{n_2+1}(t) + p_2(1-\theta) p_1 \int_0^\infty P^{(2)}_{n_2}(x,t) \mu_2(x) dx + p_2(1-\theta) q_1 \int_0^\infty P^{(2)}_{n_2+1}(x,t) \mu_2(x) dx
\end{align*}

It is assumed that initially there is no customer in the system, the server is not under vacation and the server is idle. So the initial conditions are

\[ V_0(0) = V_n(0) = 0; Q(0) = 0 \text{ and } P^{(i)}_n(0) = 0, n = 0, 1, 2, ..., j = 1, 2 \]

**GENERATING FUNCTIONS OF THE QUEUE LENGTH: THE TIME–DEPENDENT SOLUTION**

In this section, the transient solution for the above set of differential–difference equations are obtained.

**THEOREM:**

The system of differential difference equations to describe an M/G/1 queue with preemptive repeat priority service and Bernoulli schedule server vacation for the two priority queues q_1 and q_2 are respectively given by equation (3)-(6) and (7)-(11) with initial conditions given by (16) and the boundary conditions by (12) – (15). Further the generating functions of transient solution are given by equation (52) to (57).

**Proof:**

The probability generation functions defined as follows.

\[ P^{(1)}_{n_1}(x,z,t) = \sum_{n=0}^\infty z^n P^{(1)}_{n_1}(x,t) \]
A single server M/G/1 queue

\[ P_{n_1}^{(1)}(z,t) = \sum_{n_0=0}^{\infty} z^n P_{n_1}^{(1)}(t) \quad (17) \]

\[ P_{n_2}^{(2)}(x,z,t) = \sum_{n_2=0}^{\infty} z^n P_{n_2}^{(1)}(x,t) \]

\[ P_{n_2}^{(2)}(z,t) = \sum_{n_2=0}^{\infty} z^n P_{n_2}^{(1)}(t) \quad (18) \]

\[ V_{n_1}(z,t) = \sum_{n_0=0}^{\infty} z^n V_{n_1}(t) \]

\[ V_{n_2}(z,t) = \sum_{n_2=0}^{\infty} z^n V_{n_2}(t) \quad (19) \]

are convergent inside the circle given by \(|z| \leq 1\). Define the Laplace transform of the function \( f(t) \) as

\[ \overline{f}(s) = \int_{0}^{\infty} e^{-st} f(t) dt; \mathfrak{R}(s) > 0 \quad (20) \]

Taking the Laplace transforms of equations (3) – (19) and using (16), we obtain

\[ \frac{\partial}{\partial x} P_{n_1}^{(1)}(x,s) + \left[s + \lambda + \mu_1(x)\right] P_{n_1}^{(1)}(x,s) = \lambda P_{n_1}^{(1)}(x,s) \quad (21) \]

\[ \frac{\partial}{\partial x} P_{n_2}^{(1)}(x,s) + \left[s + \lambda + \mu_1(x)\right] P_{n_2}^{(1)}(x,s) = 0 \quad (22) \]

\[ \frac{\partial}{\partial x} P_{n_2}^{(2)}(x,s) + \left[s + \lambda + \mu_2(x)\right] P_{n_2}^{(2)}(x,s) = \lambda P_{n_2}^{(2)}(x,s) \quad (23) \]

\[ \frac{\partial}{\partial x} P_{0}^{(2)}(x,s) + \left[s + \lambda + \mu_2(x)\right] P_{0}^{(2)}(x,s) = 0 \quad (24) \]

\[ (s + \lambda + \beta) \overline{V_{0}}(s) = \theta \int_{0}^{\infty} P_{0}^{(1)}(x,s) \mu_1(x) dx \quad (25) \]

\[ (s + \lambda + \beta) \overline{V_{n_1}}(s) = \lambda \overline{V_{n_1-1}}(s) + \theta \int_{0}^{\infty} P_{n_1}^{(1)}(x,s) \mu_1(x) dx \quad (26) \]

\[ (s + \lambda + \beta) \overline{V_{n_2}}(s) = \theta \int_{0}^{\infty} P_{n_2}^{(1)}(x,s) \mu_2(x) dx \quad (27) \]

\[ (s + \lambda + \beta) \overline{V_{n_2}}(s) = \lambda \overline{V_{n_2-1}}(s) + \theta \int_{0}^{\infty} P_{n_2}^{(2)}(x,s) \mu_2(x) dx \quad (28) \]
\[(s + \lambda)\overline{Q}(s) = 1 + \beta \overline{V}_0(s) + (1 - \theta) \int_0^\infty P_0^{(1)}(x, s) \mu_1(x) dx + q(1 - \theta) \int_0^\infty P_0^{(2)}(x, s) \mu_2(x) dx \]

Now taking Laplace transform for the boundary conditions given by (12) to (15), we get

\[
\overline{P}_0^{(1)}(0, s) = \overline{Q}_1(s) \lambda p_1 + p_1 \beta \overline{V}_1(s) + p_1 (1 - \theta) \int_0^\infty P_0^{(1)}(x, s) \mu_1(x) dx
\]

\[
\overline{P}_n^{(1)}(0, s) = p_n \beta \overline{V}_{n+1}(s) + p_1 (1 - \theta) \int_0^\infty P_n^{(1)}(x, s) \mu_1(x) dx
\]

\[
\overline{P}_0^{(2)}(0, s) = \overline{Q}_2(s) \lambda p_2 + p_2 \beta \overline{V}_1(s) + p_2 (1 - \theta) \int_0^\infty P_0^{(2)}(x, s) \mu_2(x) dx + p_2 (1 - \theta) q \int_0^\infty P_0^{(2)}(x, s) \mu_2(x) dx
\]

\[
\overline{P}_{n_2}^{(2)}(0, s) = p_2 \beta \overline{V}_{n_2+1}(s) + p_2 (1 - \theta) \int_0^\infty P_{n_2}^{(2)}(x, s) \mu_2(x) dx + p_2 (1 - \theta) q \int_0^\infty P_{n_2+1}^{(2)}(x, s) \mu_2(x) dx
\]

We multiply equations (21) and (22) by suitable powers of \(z\), and summing it over \(n_1\) from 1 to \(\infty\) and use (17). After algebraic simplifications we get

\[
\frac{\partial}{\partial x} \overline{P}_n^{(1)}(x, z, s) + (s + \lambda - \lambda z + \mu(x)) \overline{P}_n^{(1)}(x, z, s) = 0
\]

Performing similar operations on equations (23) and (24) and using (18), we have

\[
\frac{\partial}{\partial x} \overline{P}_{n_2}^{(2)}(x, z, s) + (s + \lambda - \lambda z + \mu_2(x)) \overline{P}_{n_2}^{(2)}(x, z, s) = 0
\]

Multiplying equations (25) and (26) by suitable powers of \(z\), summing over \(n_1\) and using (19), leads to the following after simplification

\[
(s + \lambda + \beta - \lambda z) \overline{V}_n(z, s) = \theta \int_0^\infty \overline{P}_n^{(1)}(x, z, s) \mu_1(x) dx
\]

Multiplying equations (27) and (28) by suitable powers of \(z\), summing over \(n_2\) and using (19), leads to the following after simplification

\[
(s + \lambda + \beta - \lambda z) \overline{V}_{n_2}(z, s) = \theta \int_0^\infty \overline{P}_{n_2}^{(2)}(x, z, s) \mu_2(x) dx
\]
By multiplying equation (30) by $z$ and equation (31) by $z^{n+1}$ and summing over $n_1$ from 1 to $\infty$, we get equation (38) by using (17),(18) and (19) for $q_1$,

$$zP_{q_1}^{(1)}(0, z, s) = zp_1\hat{Q}(s) + \beta p_1\hat{V}(z, s) + zp_1(1-\theta)\int_0^\infty P_{q_1}^{(1)}(x, z, s)\mu_1(x)dx$$

(38)

Performing similar operations on equations (31) and (32) for $q_2$

$$zP_{q_2}^{(2)}(0, z, s) = zp_2\hat{Q}(s) + \beta p_2\hat{V}(z, s) + (q + pz)p_2(1-\theta)\int_0^\infty P_{q_2}^{(2)}(x, z, s)\mu_2(x)dx$$

(39)

Integrating equations (34) and (35) between 0 and $x$, we obtain

$$\overline{P}_{q_1}^{(1)}(x, z, s) = \overline{P}_{q_1}^{(1)}(0, z, s)e^{-(s+\lambda-\lambda z)x} \int_0^x \mu_1(t)dt$$

(40)

$$\overline{P}_{q_2}^{(2)}(x, z, s) = \overline{P}_{q_2}^{(2)}(0, z, s)e^{-(s+\lambda-\lambda z)x} \int_0^x \mu_2(t)dt$$

(41)

where $\overline{P}_{q_1}^{(1)}(0, z, s)$ and $\overline{P}_{q_2}^{(2)}(0, z, s)$ are given by equations (38) and (39).

Integrating equation (40) with respect to $x$, we have

$$\overline{P}_{q_1}^{(1)}(z, s) = \overline{P}_{q_1}^{(1)}(0, z, s) \left[ 1 - \overline{B}_1(s+\lambda-\lambda z) \right]$$

(42)

where $\overline{B}_1(s+\lambda-\lambda z) = \int_0^\infty e^{-(s+\lambda-\lambda z)x} dB_1(x)$

(43)

From equation (40) and equation (2) we obtain

$$\int_0^\infty \overline{P}_{q_1}^{(1)}(x, z, s)\mu_1(x)dx = \overline{P}_{q_1}^{(1)}(0, z, s)\overline{B}_1(s+\lambda-\lambda z)$$

(44)

Integrating equation (41) with respect to $x$, we have

$$\overline{P}_{q_2}^{(2)}(z, s) = \overline{P}_{q_2}^{(2)}(0, z, s) \left[ 1 - \overline{B}_2(s+\lambda-\lambda z) \right]$$

(45)

where $\overline{B}_2(s+\lambda-\lambda z) = \int_0^\infty e^{-(s+\lambda-\lambda z)x} dB_2(x)$

(46)

From equation (41) and equation (2) we obtain

$$\int_0^\infty \overline{P}_{q_2}^{(2)}(x, z, s)\mu_2(x)dx = \overline{P}_{q_2}^{(2)}(0, z, s)\overline{B}_2(s+\lambda-\lambda z)$$

(47)
We now substitute the value of \( \overline{V}(z, s) \) from equation (36) into equation (37), (38) and also using the of equations (44) and (47), we obtain after simplifications

\[
\overline{zP}_{q_1}^{(1)}(0, z, s) = z p_1 \lambda \overline{Q}(s) + \beta p_1 \overline{V}(z, s) + z p_1 (1 - \theta) \overline{P}_{q_1}^{(1)}(0, z, s) \overline{B}(s + \lambda + \lambda z)
\]

(48)

\[
\overline{zP}_{q_2}^{(2)}(0, z, s) = z p_2 \lambda \overline{Q}(s) + \beta p_2 \overline{V}(z, s) + (q + p z) p_2 (1 - \theta) \overline{P}_{q_2}^{(2)}(0, z, s) \overline{B}(s + \lambda + \lambda z)
\]

(49)

From (48)

\[
\overline{zP}_{q_1}^{(1)}(0, z, s)[1 - p_1 (1 - \theta) \overline{B}(s + \lambda + \lambda z)] = z p_1 \lambda \overline{Q}(s) + \beta p_1 \overline{V}(z, s)
\]

(50)

From (49)

\[
\overline{P}_{q_2}^{(2)}(0, z, s)[z - (q + p z) p_2 (1 - \theta) \overline{B}(s + \lambda + \lambda z)] = z p_2 \lambda \overline{Q}(s) + \beta p_2 \overline{V}(z, s)
\]

(51)

From (50)

\[
\overline{zP}_{q_1}^{(1)}(0, z, s)[1 - p_1 (1 - \theta) \overline{B}(s + \lambda + \lambda z)] = z p_1 \lambda \overline{Q}(s) + \beta p_1 \frac{\theta}{s + \lambda + \beta - \lambda z} \overline{P}_{q_1}^{(1)}(0, z, s) \overline{B}(s + \lambda + \lambda z)
\]

\[
\overline{zP}_{q_1}^{(1)}(0, z, s)[1 - p_1 (1 - \theta) \overline{B}(s + \lambda + \lambda z)] - \beta p_1 \frac{\theta}{s + \lambda + \beta - \lambda z} \overline{P}_{q_1}^{(1)}(0, z, s) \overline{B}(s + \lambda + \lambda z) = z p_1 \lambda \overline{Q}(s)
\]

From (51)

\[
\overline{P}_{q_2}^{(2)}(0, z, s)[z - (q + p z) p_2 (1 - \theta) \overline{B}(s + \lambda + \lambda z)] = z p_2 \lambda \overline{Q}(s) + \beta p_2 \frac{\theta}{s + \lambda + \beta - \lambda z} \overline{P}_{q_2}^{(2)}(0, z, s) \overline{B}(s + \lambda + \lambda z)
\]

\[
\overline{P}_{q_2}^{(2)}(0, z, s)[z - (q + p z) p_2 (1 - \theta) \overline{B}(s + \lambda + \lambda z)] - \beta p_2 \frac{\theta}{s + \lambda + \beta - \lambda z} \overline{P}_{q_2}^{(2)}(0, z, s) \overline{B}(s + \lambda + \lambda z) = z p_2 \lambda \overline{Q}(s)
\]
A single server M/G/1 queue

\[
\overline{P}_{q_1}^{(2)}(0, z, s) = \frac{zp_2 \lambda \overline{Q}_2(s)(s + \lambda + \beta - \lambda z)}{(s + \lambda + \beta - \lambda z)z - p_2 \overline{B}_2(s + \lambda + \beta - \lambda z)[(q + pz)(s + \lambda + \beta - \lambda z)(1-\theta) + \beta \theta]}
\]

From (36)

\[
\overline{V}_{q_1}(z, s) = \frac{\theta}{s + \lambda + \beta - \lambda z}\int_0^\infty \overline{P}_{q_1}^{(1)}(x, z, s) \mu_1(x) dx = \frac{\theta}{s + \lambda + \beta - \lambda z} \overline{P}_{q_1}^{(1)}(0, z, s) \overline{B}_1(s + \lambda - \lambda z)
\]

From (37)

\[
\overline{V}_{q_2}(z, s) = \frac{\theta}{s + \lambda + \beta - \lambda z}\int_0^\infty \overline{P}_{q_2}^{(2)}(x, z, s) \mu_2(x) dx = \frac{\theta}{s + \lambda + \beta - \lambda z} \overline{P}_{q_2}^{(2)}(0, z, s) \overline{B}_2(s + \lambda - \lambda z)
\]

Using (50) in (42), we get

\[
\overline{P}_{q_1}^{(1)}(z, s) = \frac{zp_1 \lambda \overline{Q}(s) + \beta p_1 \overline{V}(z, s)}{z - zp_1(1-\theta) \overline{B}_1(s + \lambda - \lambda z)} \left[1 - \overline{B}_1(s + \lambda - \lambda z)\right]
\]

Using (51) in (45), we obtain

\[
\overline{P}_{q_2}^{(2)}(z, s) = \frac{zp_2 \lambda \overline{Q}(s) + \beta p_2 \overline{V}(z, s)}{z - (q + pz)p_2(1-\theta) \overline{B}_2(s + \lambda - \lambda z)} \left[1 - \overline{B}_2(s + \lambda - \lambda z)\right]
\]

The steady state results:

In this section, we shall derive the steady state probability distribution for the proposed queueing model. To define the steady state probabilities, the argument t is suppressed wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property

\[
\lim_{s \to 0} sf(s) = \lim_{t \to \infty} f(t)
\]

In order to determine \(\overline{P}_{q_1}^{(1)}(z, s), \overline{P}_{q_2}^{(2)}(z, s), \overline{V}_{q_1}(z, s)\) and \(\overline{V}_{q_2}(z, s)\) completely, we have yet to determine unknown \(Q_1\) and \(Q_2\) respectively which appears in the numerators of the right hand sides of equations (42), (45), (54), (55), (56) and (57)
For that purpose, we shall use the normalizing condition for $q_1$ and $q_2$ separately.

**Theorem:**

The steady state probability distribution for an M/G/1 queue with essential service following general distribution subject to optional server vacation based on Bernoulli schedule with single vacation policy and preemptive repeat priority service are given by

$$P_{q_1}^{(1)}(1) = \frac{p_1 \lambda Q_1 E(v_1)}{1 - p_1}$$

$$P_{q_2}^{(2)}(1) = \frac{p_2 \lambda Q_2 E(v_2)}{1 - p_2}$$

$$V_{q_1}(1) = \frac{\theta p_1 \lambda Q_1 [1 + \lambda E(v_1)]}{(\beta - \lambda) - p_1(1-\theta)(\beta - \lambda) - p_1 \lambda \beta E(v_1)}$$

$$V_{q_2}(1) = \frac{\theta p_2 \lambda Q_2 [1 + \lambda E(v_2)]}{(\beta - \lambda) - p_2(1-\theta)(\beta - \lambda) - p_2 \lambda \beta E(v_2)}$$

$$Q_1 = \frac{(1-p_1)[(\beta - \lambda) - p_1(1-\theta)(\beta - \lambda) - p_1 \lambda \beta E(v_1)]}{[1 - p_1(1-\lambda E(v_1))][(\beta - \lambda) - p_1(1-\theta)(\beta - \lambda) - p_1 \lambda \beta E(v_1)] + (1-p_1)p_2 \lambda(1+\lambda E(v_1))}$$

$$Q_2 = \frac{(1-p_2)[(\beta - \lambda) - p_2(1-\theta)(\beta - \lambda) - p_2 \lambda \beta E(v_2)]}{[1 - p_2(1-\lambda E(v_2))][(\beta - \lambda) - p_2(1-\theta)(\beta - \lambda) - p_2 \lambda \beta E(v_2)] + (1-p_2)p_2 \lambda(1+\lambda E(v_2))}$$

where $P_{q_1}^{(1)}$, $P_{q_2}^{(2)}$, $V_{q_1}(1)$, $V_{q_2}(1)$, $Q_1$ and $Q_2$ are the steady state probabilities that the server is providing essential service, server under vacation, server attending pre-emptive repeat priority service and server being idle respectively independent of the number of customers in the queue.

**Proof:**

Multiplying both sides of equations (42),(45), (54) and (55) by $s$, taking limit as $s \rightarrow 0$, applying property (58) and simplifying, we obtain

$$P_{q_1}^{(1)}(z) = \frac{zp_1 \lambda Q_1 (\lambda + \beta - \lambda z)}{z(\lambda + \beta - \lambda z) - p_1 B_1(\lambda - \lambda z)[z(\lambda + \beta - \lambda z)(1-\theta) + \beta \theta]} \left[1 - \frac{B_z(\lambda - \lambda z)}{\lambda - \lambda z}\right]$$

(65)
A single server M/G/1 queue

\[ P_{q_2}^{(2)}(z) = \frac{zp_2\lambda Q_2(\lambda + \beta - \lambda z)}{z(\lambda + \beta - \lambda z) - p_2\overline{B}_2(\lambda - \lambda z)[(q + p_2)(\lambda + \beta - \lambda z)(1 - \theta) + \beta \theta]} \left[ \frac{1 - \overline{B}_2(\lambda - \lambda z)}{\lambda - \lambda z} \right] \]

\[ V_{q_2}(z) = \frac{\theta}{(\lambda + \beta - \lambda z)} \left[ z - p_1\overline{B}_1(\lambda - \lambda z) \left[ z(\lambda + \beta - \lambda z)(1 - \theta) + \beta \theta \right] \overline{B}_1(\lambda - \lambda z) \right] \]

\[ V_{q_2}(z) = \frac{\theta}{(\lambda + \beta - \lambda z)} \left[ z - p_2\overline{B}_2(\lambda - \lambda z) \left[ (q + p_2)(\lambda + \beta - \lambda z)(1 - \theta) + \beta \theta \right] \overline{B}_2(\lambda - \lambda z) \right] \]

For \( P_{q_2}^{(1)}(z) \)

\[ Nr = p_1\lambda Q_1 z(\lambda + \beta - \lambda z) \left[ 1 - \overline{B}_1(\lambda - \lambda z) \right] \]

\[ Nr = p_1\lambda Q_1 \left[ 1(\lambda + \beta - \lambda z) \left[ 1 - \overline{B}_1(\lambda - \lambda z) \right] + z(-\lambda) \left[ 1 - \overline{B}_1(\lambda - \lambda z) \right] + z(\lambda + \beta - \lambda z) \left[ -\overline{B}_1(\lambda - \lambda z) \right] \right] \]

\[ \overline{B}_1(0) = 1; \overline{B}_1^i(0) = E(v_i); \overline{B}_1^i(0) = E(v_i^2); i = 1, 2 \]

\[ Nr^r(1) = p_1\lambda Q_1 \left[ 1. \beta E(v_i) \right] \left[ (-\lambda) \right] = -p_1\lambda^2 Q_1 \beta E(v_i) \]

\[ Dr = \left\{ \left[ z(\lambda + \beta - \lambda z) - p_1\overline{B}_1(\lambda - \lambda z) \left[ z(\lambda + \beta - \lambda z)(1 - \theta) + \beta \theta \right] \right] (\lambda - \lambda z) \right\} \]

\[ Dr = \left\{ z(\lambda + \beta - \lambda z)(\lambda - \lambda z) - p_1\overline{B}_1(\lambda - \lambda z) z(\lambda + \beta - \lambda z)(\lambda - \lambda z)(1 - \theta) - p_1\overline{B}_1(\lambda - \lambda z) \beta \theta (\lambda - \lambda z) \right\} \]

\[ Dr^r(1) = 1(\lambda + \beta - \lambda z)(\lambda - \lambda z) + z(-\lambda)(\lambda - \lambda z) + z(\lambda + \beta - \lambda z)(-\lambda) \]

\[ - p_1(1 - \theta) \left[ \overline{B}_1(\lambda - \lambda z)(\lambda - \lambda z)(-\lambda) + \overline{B}_1(\lambda - \lambda z)(\lambda + \beta - \lambda z) \right. \left. + \overline{B}_1(\lambda - \lambda z)(\lambda + \beta - \lambda z)(\lambda - \lambda z) \right] \]

\[ + \overline{B}_1(\lambda - \lambda z)(\lambda - \lambda z)(\lambda - \lambda z) \]

\[ Dr^r(1) = 1(\beta)(-\lambda) - p_1(1 - \theta) \left[ (-\lambda \beta) \right] + p_1 \beta \theta \left[ (\lambda) \right] \]
\[ Dr' (1) = -\beta \lambda (1 - p_1) \]
\[ P^{(1)}_{q_1} (1) = \frac{p_1 \lambda Q E (v_i)}{1 - p_1} \]

For \( P^{(2)}_{q_2} (z) \)
\[ Nr = p_2 \lambda Q z \left( \lambda + \beta - \lambda z \right) \left[ 1 - B_2 (\lambda - \lambda z) \right] \]

\[ Nr' = p_2 \lambda Q z \left( \lambda + \beta - \lambda z \right) \left[ 1 - B_2 (\lambda - \lambda z) \right] + z(-\lambda) \left[ 1 - B_2 (\lambda - \lambda z) \right] + z(\lambda + \beta - \lambda z) \left[ -B_2 (\lambda - \lambda z) \right] (-\lambda) \]

\[ B_i (0) = 1; -\overline{B}_i (0) = E(v_i); i = 1, 2 \]
\[ Nr' (1) = p_2 \lambda Q_2 \{ 1, \beta E(v_i) \}(-\lambda) = -p_2 \lambda^2 Q_2 \beta E(v_i) \]
\[ Dr = \left\{ z(\lambda + \beta - \lambda z) - p_2 \overline{B}_2 (\lambda - \lambda z) \left[ (q + pz)(\lambda + \beta - \lambda z)(1 - \theta) + \beta \theta \right] \right\} (\lambda - \lambda z) \]

\[ Dr' = 1, (\lambda + \beta - \lambda z)(\lambda - \lambda z) + z(-\lambda)(\lambda - \lambda z) + z(\lambda + \beta - \lambda z)(-\lambda) \]
\[ - p_2 (1 - \theta) \left\{ (\lambda - \lambda z) \overline{B}_2 (\lambda - \lambda z)(q + pz)(-\lambda) + \overline{B}_2 (\lambda - \lambda z)(p)(\lambda - \lambda z)(\lambda + \beta - \lambda z) \right\} \]
\[ + \overline{B}_2 (\lambda - \lambda z)(q + pz)(\lambda + \beta - \lambda z)(-\lambda) + (\overline{B}_2 (\lambda - \lambda z)(q + pz)(\lambda + \beta - \lambda z)(\lambda - \lambda z) \right\} \]
\[ - p_2 \beta \theta \left\{ \overline{B}_2 (\lambda - \lambda z)(-\lambda) + (\overline{B}_2 (\lambda - \lambda z)(\lambda - \lambda z) \right\} \]

\[ Dr' (1) = -\lambda \beta + p_2 \lambda \left[ 1 - \theta (1 + \beta) \right] + p_2 \lambda \beta \theta \]
\[ Dr' (1) = -\lambda \beta (1 - p_2) \]
\[ P^{(2)}_{q_2} (1) = \frac{p_2 \lambda Q_2 E (v_2)}{1 - p_2} \]

Proceeding on similar lines, we get
\[ V_{q_1} (1) = \frac{\theta p_1 \lambda Q_1 \left[ 1 + \lambda E (v_i) \right]}{(\beta - \lambda) - p_1 \left( 1 - \theta \right)(\beta - \lambda) - p_1 \lambda \beta E (v_i)} \]
\[ V_{q_1} (1) = \frac{\theta p_2 \lambda Q_2 \left[ 1 + \lambda E (v_2) \right]}{(\beta - \lambda) - p_2 \left( 1 - \theta \right)(p \beta - \lambda) - p_2 \lambda \beta E (v_2)} \]
A single server $M/G/1$ queue

Normalized condition for high priority queue $q_1$ is given by

$$ p^{(1)}_n + V_{q_1} (1) + Q_1 = 1 $$

$$ Q_1 \left[ \frac{p_1 \lambda E(v_i)}{1 - p_1} + \frac{\theta p_1 \lambda Q_1 [1 + \lambda E(v_i)]}{(\beta - \lambda) - p_1 (1 - \theta)(\beta - \lambda) - p_1 \lambda \beta E(v_i)} \right] + Q_1 = 1 $$

$$ Q_1 = \frac{(1 - p_1)\left[(\beta - \lambda) - p_1 (1 - \theta)(\beta - \lambda) - p_1 \lambda \beta E(v_i)\right]}{[1 - p_1 (1 - \lambda E(v_i))][\beta - \lambda - p_1 (1 - \theta)(\beta - \lambda) - p_1 \lambda \beta E(v_i)] + (1 - p_1) p_1 \theta \lambda (1 + \lambda E(v_i))} \quad (70) $$

$$ \rho_1 = 1 - Q_1 $$

$$ \rho_1 = 1 - \frac{(1 - p_1)\left[(\beta - \lambda) - p_1 (1 - \theta)(\beta - \lambda) - p_1 \lambda \beta E(v_i)\right]}{[1 - p_1 (1 - \lambda E(v_i))][\beta - \lambda - p_1 (1 - \theta)(\beta - \lambda) - p_1 \lambda \beta E(v_i)] + (1 - p_1) p_1 \theta \lambda (1 + \lambda E(v_i))} $$

$$ \rho_1 = \frac{[p_1 \lambda E(v_i)]\left[(\beta - \lambda) - p_1 (1 - \theta)(\beta - \lambda) - p_1 \lambda \beta E(v_i)\right]}{[1 - p_1 (1 - \lambda E(v_i))][\beta - \lambda - p_1 (1 - \theta)(\beta - \lambda) - p_1 \lambda \beta E(v_i)] + (1 - p_1) p_1 \theta \lambda (1 + \lambda E(v_i))} $$

$$ \rho_1 = \frac{[p_1 \lambda E(v_i)]\left[(\beta - \lambda) - p_1 (1 - \theta)(\beta - \lambda) - p_1 \lambda \beta E(v_i)\right]}{[1 - p_1 (1 - \lambda E(v_i))][\beta - \lambda - p_1 (1 - \theta)(\beta - \lambda) - p_1 \lambda \beta E(v_i)] + (1 - p_1) p_1 \theta \lambda (1 + \lambda E(v_i))} \quad (71) $$

Normalized condition for low priority queue $q_2$ is given by

$$ p^{(2)}_n (1) + V_{q_2} (1) + Q_2 = 1 $$

$$ Q_2 \left[ \frac{p_2 \lambda E(v_2)}{1 - p_2} + \frac{\theta p_2 \lambda Q_2 [1 + \lambda E(v_2)]}{(\beta - \lambda) - p_2 (1 - \theta)(\beta - \lambda) - p_2 \lambda \beta E(v_2)} \right] + Q_2 = 1 $$

$$ Q_2 = \frac{(1 - p_2)\left[(\beta - \lambda) - p_2 (1 - \theta)(\beta - \lambda) - p_2 \lambda \beta E(v_2)\right]}{[1 - p_2 (1 - \lambda E(v_2))][\beta - \lambda - p_2 (1 - \theta)(\beta - \lambda) - p_2 \lambda \beta E(v_2)] + (1 - p_2) p_2 \theta \lambda (1 + \lambda E(v_2))} \quad (73) $$
\[ \rho_2 = 1 - Q_2 \]

\[ \rho_2 = 1 - \frac{(1 - p_2) \left[ (\beta - \lambda) - p_2 (1 - \theta)(p \beta - \lambda) - p_2 \lambda \beta E(v_2) \right]}{1 - p_2 (1 - \lambda E(v_2)) \left[ (\beta - \lambda) - p_2 (1 - \theta)(p \beta - \lambda) - p_2 \lambda \beta E(v_2) \right] + (1 - p_2) P_2 \lambda (1 + \lambda E(v_2))} \]

where \( \rho_1 < 1 \) and \( \rho_2 < 1 \) are the stability conditions under which the steady state exists for \( q_1 \) and \( q_2 \) respectively. Equations (70),(73) also gives the probability that the server is idle. Substituting \( Q_1 \) and \( Q_2 \) in (69) and (72), we have completely and explicitly determined \( W_{q_1}(z) \) and \( W_{q_2}(z) \), the probability generating function of the queue size where \( W_{q_i}(z) = P_{q_i}^{(1)}(z) + V_{q_i}(z) + Q_i \) and \( W_{q_i}(z) = P_{q_i}^{(2)}(z) + V_{q_i}(z) + Q_i \).

The average queue size and the average waiting time for high priority queue \( q_1 \):

Let \( L_{q_1} \) denote the mean number of customers in the queue \( q_1 \) under the steady state.

Then \( L_{q_1} = \frac{d}{dz} P_{q_1}(z) \) at \( z = 1 \). Since \( P_{q_1}(z) \) takes indeterminate form \( \frac{0}{0} \) at \( z = 1 \)

where \( P_{q_1}(z) = P_{q_1}^{(1)}(z) + V_{q_1}(z) \)

\[ = \frac{z p_1 \lambda Q_1 (\lambda + \beta - \lambda z)}{z(\lambda + \beta - \lambda z) - p_1 B_1(\lambda - \lambda z) \left[ z(\lambda + \beta - \lambda z)(1 - \theta) + \beta \theta \right]} \left[ 1 - \bar{B}_1(\lambda - \lambda z) \right] \]

\[ + \frac{z p_1 \lambda Q_1}{(\lambda + \beta - \lambda z) z - p_1 B_1(\lambda - \lambda z) \left[ z(\lambda + \beta - \lambda z)(1 - \theta) + \beta \theta \right]} \bar{B}_1(\lambda - \lambda z) \]

\[ P_{q_1}(z) = \frac{z p_1 \lambda Q_1 (\lambda + \beta - \lambda z)}{(\lambda - \lambda z) \left[ z(\lambda + \beta - \lambda z) - p_1 B_1(\lambda - \lambda z) \right]} \left[ 1 - \bar{B}_1(\lambda - \lambda z) \right] + (\lambda - \lambda z) z \theta p_1 \lambda Q_1 \bar{B}_1(\lambda - \lambda z) \]

\[ (\lambda - \lambda z) \left[ z(\lambda + \beta - \lambda z) - p_1 B_1(\lambda - \lambda z) \right] \left[ z(\lambda + \beta - \lambda z)(1 - \theta) + \beta \theta \right] \]

We use the following well-known result in queueing theory (Kashyap and Chaudhry (3)).

This is applied when \( P_{q_1}(z) \) is indeterminate of the form \( \frac{0}{0} \)
A single server M/G/1 queue

\[
L_q = \lim_{z \to 1} \frac{d}{dz} F_q(z) = \lim_{z \to 1} \frac{D(z) N'(z) - N'(z) D'(z)}{2(D(z))^2} = \frac{D'(1) N'(1) - N'(1) D'(1)}{2(D(1))^2} \tag{75}
\]

\[
N_r = z p_i \lambda Q_i \left[ 1 - B_i(\lambda - \lambda z) \right] + (\lambda - \lambda z) \left\{ \theta z p_i \lambda Q_i B_i(\lambda - \lambda z) \right\}
\]

\[
N_r' = p_i \lambda Q_i \left\{ (1 + \beta - \lambda z) \left[ 1 - B_i(\lambda - \lambda z) \right] + z(\lambda - \lambda z) \left[ -B_i(\lambda - \lambda z)(-\lambda) \right] \right\}
\]

\[
+ \theta p_i \lambda Q_i \left\{ (1 - \lambda z) B_i(\lambda - \lambda z) + z(\lambda + \beta - \lambda z) B_i(\lambda - \lambda z)(-\lambda) \right\}
\]

\[
N_r' \left( L \right) = -p_i \lambda^2 Q_i \left[ \beta E(v_i) + \theta \right]
\]

\[
N_r' = p_i \lambda Q_i \left\{ (-\lambda) \left[ 1 - B_i(\lambda - \lambda z) \right] + (\lambda + \beta - \lambda z) \left[ -B_i(\lambda - \lambda z)(-\lambda) \right] - \lambda \left[ 1 - B_i(\lambda - \lambda z) \right] - \lambda z \left[ -B_i(\lambda - \lambda z)(-\lambda) \right] \right\}
\]

\[
+ p_i \lambda^2 Q_i \left\{ (\lambda + \beta - \lambda z) B_i(\lambda - \lambda z) + z(\lambda + \beta - \lambda z) B_i(\lambda - \lambda z)(-\lambda) \right\}
\]

\[
+ \theta p_i \lambda Q_i \left\{ (-\lambda) B_i(\lambda - \lambda z) + (\lambda - \lambda z) B_i(\lambda - \lambda z)(-\lambda) + (\lambda)(1 - B_i(\lambda - \lambda z) + z B_i(\lambda - \lambda z)(-\lambda)) \right\}
\]

\[
- \theta p_i \lambda^2 Q_i \left\{ (-\lambda) B_i(\lambda - \lambda z) + z(\lambda - \lambda z) B_i(\lambda - \lambda z)(-\lambda) \right\}
\]

\[
N_r' \left( L \right) = -p_i \lambda^2 Q_i \beta E(v_i) + p_i \lambda^3 Q_i \beta E(v_i) - p_i \lambda^2 Q_i \beta E(v_i) - p_i \lambda^3 Q_i \beta E(v_i) - p_i \lambda^3 Q_i \beta E(v_i)
\]

\[
- \theta p_i \lambda^2 Q_i - \theta p_i \lambda^2 Q_i - \theta p_i \lambda^3 Q_i E(v_i) - \theta p_i \lambda^3 Q_i E(v_i)
\]

\[
= -2 p_i \lambda^2 Q_i \beta E(v_i) + 2 p_i \lambda^3 Q_i E(v_i) - p_i \lambda^3 Q_i \beta E(v_i) - 2 \theta p_i \lambda^3 Q_i E(v_i)
\]

\[
= -2 p_i \lambda^2 Q_i E(v_i) \left[ \lambda - \beta \right] - p_i \lambda^3 Q_i \beta E(v_i) - 2 \theta p_i \lambda^2 Q_i E(v_i)
\]

\[
Dr = \left( \lambda - \lambda z \right) \left\{ (\lambda + \beta - \lambda z) z - p_i B_i(\lambda - \lambda z) \left[ z(\lambda + \beta - \lambda z)(1 - \theta) + \beta \theta \right] \right\}
\]

\[
Dr = \left( \lambda - \lambda z \right) \left\{ (\lambda + \beta - \lambda z) z - p_i B_i(\lambda - \lambda z) z(\lambda + \beta - \lambda z)(1 - \theta) - p_i \beta \theta B_i(\lambda - \lambda z) \right\}
\]
\[ D_r = -\lambda z (\beta + \lambda - \lambda z) + (\lambda - \lambda z)(\beta + \lambda - \lambda z) - \lambda z (\lambda - \lambda z) \]
\[ - p_1 (1 - \theta) \left\{ -\lambda \overline{B}_1 (\lambda - \lambda z) z (\beta + \lambda - \lambda z) - \lambda (\lambda - \lambda z) \overline{B}_1 (\lambda - \lambda z) z (\beta + \lambda - \lambda z) \right\} \]
\[ - p_1 \beta \theta \left\{ -\lambda \overline{B}_1 (\lambda - \lambda z) - \lambda (\lambda - \lambda z) \overline{B}_1 (\lambda - \lambda z) \right\} \]
\[ D_r(1) = -\lambda \beta (1 - p_1) \]
\[ D_r = (-\lambda)(\lambda + \beta - \lambda z) - \lambda z (-\lambda) + (\lambda - \lambda z)(-\lambda) - \lambda (-\lambda) z - \lambda (\lambda - \lambda z) \]
\[ - p_1 (1 - \theta) \left\{ -\lambda \overline{B}_1 (\lambda - \lambda z) z (-\lambda) + (\lambda - \lambda z) (\beta + \lambda - \lambda z) \right\} \]
\[ - p_1 (1 - \theta) \left\{ -\lambda (\lambda - \lambda z) \overline{B}_1 (\lambda - \lambda z) z (-\lambda) + (\lambda - \lambda z) (\beta + \lambda - \lambda z) \right\} \]
\[ \overline{B}_1(0) = 1; \overline{B}_1^\prime(0) = E(v_i); \overline{B}_1(0) = E(v_i^2) \]
\[ D_r(1) = -2\lambda \beta + 2\lambda^2 + 2\lambda^2 p_1 \beta E(v_i) + 2\lambda p_1 \beta (1 - \theta) - 2\lambda^2 p_1 (1 - \theta) \]

Further we find the average system size \( L \) is found using Little’s formula. Thus
\[ L = L_{\eta} + \rho_1 \]
Where \( L_{\eta} \) has been found in equation (75) and \( \rho_1 \) is obtained from equation (71).

The average queue size and the average waiting time for low priority queue \( q_z \):
\[ P_{q_z}(z) = P_{q_z}^{(2)}(z) + V_{q_z}(z) \]
\[ = \frac{zp_2 \lambda Q_z(\lambda + \beta - \lambda z)}{z(\lambda + \beta - \lambda z) - p_2 \overline{B}_z(\lambda - \lambda z) \left[ z(1 - \theta) - \beta \theta \right] \left[ \frac{1 - \overline{B}_z(\lambda - \lambda z)}{\lambda - \lambda z} \right]} \]
A single server M/G/1 queue

\[
\frac{\theta}{(\lambda + \beta - \lambda z)} + \frac{zp_2 \lambda Q_2}{z - p_2 B_2(\lambda - \lambda z)(1 - \theta - \beta \theta)} B_2(\lambda - \lambda z)
\]

\[P_{q_2}(z) = \frac{zp_2 \lambda Q_2(\lambda + \beta - \lambda z)\left[1 - B_2(\lambda - \lambda z)\right] + (\lambda - \lambda z)\left[\theta zp_2 \lambda Q_2 B_2(\lambda - \lambda z)\right]}{z(\lambda + \beta - \lambda z) - p_2 B_2(\lambda - \lambda z)\left[(q + pz)(1 - \theta - \beta \theta)\right]} \]

The following well-known result in queueing theory (Kashyap and Chaudhry (3)) is used.

This is applied when \( P_{q_2}(z) \) is indeterminate of the form \( \frac{0}{0} \)

\[
L_{q_2} = \lim_{z \to 1} \frac{d}{dz} P_{q_2}(z) = \lim_{z \to 1} \frac{D(z)N'(z) - N(z)D'(z)}{2(D(z))^2} = \frac{D'(1)N'(1) - N'(1)D'(1)}{2D'(1)^2} \tag{77}
\]

\[
Nr = zp_2 \lambda Q_2(\lambda + \beta - \lambda z)\left[1 - B_2(\lambda - \lambda z)\right] + (\lambda - \lambda z)\left[\theta zp_2 \lambda Q_2 B_2(\lambda - \lambda z)\right]
\]

\[
Nr' = p_2 \lambda Q_2\left\{(\lambda + \beta - \lambda z)\left[1 - B_2(\lambda - \lambda z)\right] + z(\lambda + \beta - \lambda z)\right\} + (\lambda - \lambda z)\left[ -B_2(\lambda - \lambda z)(\lambda) \right] + \theta p_2 \lambda Q_2 B_2(\lambda - \lambda z) + z(\lambda - \lambda z)B_2(\lambda - \lambda z)(\lambda)
\]

\[
Nr'(1) = -p_2 \lambda^2 Q_2 \left[ \beta E(v_2) + \theta \right]
\]

\[
Nr^- = p_2 \lambda Q_2\left\{(\lambda + \beta - \lambda z)\left[1 - B_2(\lambda - \lambda z)\right] + (\lambda + \beta - \lambda z)\right\} - z(\lambda + \beta - \lambda z)\left[-B_2(\lambda - \lambda z)(\lambda)\right] + \theta p_2 \lambda Q_2 B_2(\lambda - \lambda z) + z(\lambda - \lambda z)B_2(\lambda - \lambda z)(\lambda)
\]

\[
Nr'(1) = -p_2 \lambda^2 Q_2 \left[ \beta E(v_2) + \theta \right] - p_2 \lambda^2 Q_2 \beta E(v_2) - p_2 \lambda^2 Q_2 \beta E(v_2) - p_2 \lambda^3 Q_2 \beta E(v_2) - \theta p_2 \lambda^2 Q_2 - \theta p_2 \lambda^2 Q_2 - \theta p_2 \lambda^3 Q_2 E(v_2) - \theta p_2 \lambda^3 Q_2 E(v_2)
\]

\[
= -2p_2 \lambda^3 Q_2 \beta E(v_2) + 2p_2 \lambda^3 Q_2 E(v_2) - p_2 \lambda^3 Q_2 \beta E(v_2) - 2\theta p_2 \lambda^3 Q_2 - 2\theta p_2 \lambda^3 Q_2 E(v_2)
\]
\[ = -2p_2\lambda^2 Q_E(v_2)[\lambda - \beta] - p_2\lambda^3 Q_2\beta E(v_2^2) - 2\theta p_2\lambda^2 Q_2[1 + \lambda E(v_2)] \]

\[ Dr = (\lambda - \lambda z)\{(\lambda + \beta - \lambda z)z - p_2 \overline{B}_z(\lambda - \lambda z)[(q + p z)(1 - \theta) - \beta \theta]\} \]

\[ Dr = (\lambda - \lambda z)(\lambda + \beta - \lambda z)z - p_2(\lambda - \lambda z)\overline{B}_z(\lambda - \lambda z)(q + p z)(1 - \theta) - \beta \theta p_2(\lambda - \lambda z)\overline{B}_z(\lambda - \lambda z) \]

\[ Dr = (-\lambda)(\lambda + \beta - \lambda z)z + (\lambda - \lambda z)(-\lambda)z + (\lambda - \lambda z)(\lambda + \beta - \lambda z) \]

\[ = -p_2(1 - \theta)\left\{(-\lambda)\overline{B}_z(\lambda - \lambda z)(q + p z) + (\lambda - \lambda z)\overline{B}_z(\lambda - \lambda z)(p) + (\lambda - \lambda z)\overline{B}_z(\lambda - \lambda z)(q + p z)(-\lambda)\right\} \]

\[ = -\beta \theta p_2\left\{(-\lambda)\overline{B}_z(\lambda - \lambda z) + (\lambda - \lambda z)\overline{B}_z(\lambda - \lambda z)(-\lambda)\right\} \]

\[ Dr(1) = \lambda \left\{ p_2[1 - \theta(1 + \beta)] - \beta \right\} \]

Differentiate with respect to z

\[ Dr = (-\lambda)(\lambda + \beta - \lambda z)z + (\lambda - \lambda z)(-\lambda)z + (\lambda - \lambda z)(\lambda + \beta - \lambda z) \]

\[ = -p_2(1 - \theta)\left\{(-\lambda)\overline{B}_z(\lambda - \lambda z)(p) + (\lambda - \lambda z)\overline{B}_z(\lambda - \lambda z)(q + p z)(-\lambda)\right\} \]

\[ = -\beta \theta p_2\left\{(-\lambda)\overline{B}_z(\lambda - \lambda z) + (\lambda - \lambda z)\overline{B}_z(\lambda - \lambda z)(-\lambda)\right\} \]

\[ Dr = (-\lambda)(\lambda + \beta - \lambda z)z + (\lambda - \lambda z)(-\lambda)z + (\lambda - \lambda z)(\lambda + \beta - \lambda z) \]

\[ = -p_2(1 - \theta)\left\{(-\lambda)\overline{B}_z(\lambda - \lambda z)(p) + (\lambda - \lambda z)\overline{B}_z(\lambda - \lambda z)(q + p z)(-\lambda)\right\} \]

\[ = -\beta \theta p_2\left\{(-\lambda)\overline{B}_z(\lambda - \lambda z) + (\lambda - \lambda z)\overline{B}_z(\lambda - \lambda z)(-\lambda)\right\} \]
A single server M/G/1 queue

\[Dr^*(1) = -\beta \lambda + \lambda^2 - \beta \lambda + \lambda^2 + p_2 (1-\theta) \lambda^2 E(v_2) + p_2 (1-\theta) p \lambda + p_2 (1-\theta) \lambda^2 E(v_2) + p_2 (1-\theta) p \lambda - p_2 \beta \theta \lambda^2 E(v_2) - p_2 \beta \theta \lambda^2 E(v_2)\]

\[Dr^*(1) = 2\lambda (\lambda - \beta) + 2 p_2 \lambda^2 E(v_2) [1 - \theta (1 + \beta)] + 2 p_2 p \lambda (1 - \theta)\]

Further the average system size \(L\) is formed using Little’s formula.

Thus we have \(L = L_{q_2} + \rho_2\) \hspace{1cm} (78)

where \(L_{q_2}\) has been found in equation (77) and \(\rho_2\) is obtained from equation (74).

The Mean waiting time:

Let \(W_{q_1}, W_{q_2}\) and \(W\) denote the mean waiting time in the queue \(q_1\) and \(q_2\) and the system respectively. Then using Little’s formula, we obtain

For high priority queue

\[W_{q_1} = \frac{L_{q_1}}{\lambda}\]

\[W = \frac{L}{\lambda}\]

For low priority queue

\[W_{q_2} = \frac{L_{q_2}}{\lambda}\]

\[W = \frac{L}{\lambda}\]

Where \(L_{q_1}\) and \(L_{q_2}\) and \(L\) have been found in equations (75), (77), (76) and (78).
NUMERICAL ILLUSTRATION:

Table 1: High Priority queue \( q_1 \): Let \( \mu_1 = 5; \lambda = 2; \theta = 0.2; \beta = 9 \)

<table>
<thead>
<tr>
<th>( p_1 )</th>
<th>( Q_1 )</th>
<th>( \rho_1 )</th>
<th>( L_{q_1} )</th>
<th>( L )</th>
<th>( w_{q_1} )</th>
<th>( w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.949</td>
<td>0.051</td>
<td>0.021</td>
<td>0.072</td>
<td>0.011</td>
<td>0.036</td>
</tr>
<tr>
<td>0.2</td>
<td>0.891</td>
<td>0.109</td>
<td>0.048</td>
<td>0.156</td>
<td>0.024</td>
<td>0.078</td>
</tr>
<tr>
<td>0.3</td>
<td>0.826</td>
<td>0.174</td>
<td>0.083</td>
<td>0.257</td>
<td>0.041</td>
<td>0.129</td>
</tr>
<tr>
<td>0.4</td>
<td>0.75</td>
<td>0.25</td>
<td>0.13</td>
<td>0.38</td>
<td>0.065</td>
<td>0.19</td>
</tr>
<tr>
<td>0.5</td>
<td>0.659</td>
<td>0.341</td>
<td>0.195</td>
<td>0.536</td>
<td>0.098</td>
<td>0.268</td>
</tr>
<tr>
<td>0.6</td>
<td>0.547</td>
<td>0.453</td>
<td>0.288</td>
<td>0.741</td>
<td>0.44</td>
<td>0.37</td>
</tr>
<tr>
<td>0.7</td>
<td>0.38</td>
<td>0.62</td>
<td>0.391</td>
<td>1.011</td>
<td>0.195</td>
<td>0.506</td>
</tr>
<tr>
<td>0.8</td>
<td>0.738</td>
<td>0.262</td>
<td>1.836</td>
<td>2.098</td>
<td>0.918</td>
<td>1.049</td>
</tr>
<tr>
<td>0.9</td>
<td>0.238</td>
<td>0.762</td>
<td>2.494</td>
<td>3.256</td>
<td>1.247</td>
<td>1.628</td>
</tr>
</tbody>
</table>

Table 2: Low Priority queue \( q_2 \): Let \( \mu_2 = 3; \lambda = 2; \theta = 0.2; \beta = 9; p = 1 \)

<table>
<thead>
<tr>
<th>( p_2 )</th>
<th>( Q_2 )</th>
<th>( \rho_2 )</th>
<th>( L_{q_2} )</th>
<th>( L )</th>
<th>( w_{q_2} )</th>
<th>( w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.921</td>
<td>0.079</td>
<td>0.054</td>
<td>0.132</td>
<td>0.027</td>
<td>0.066</td>
</tr>
<tr>
<td>0.2</td>
<td>0.837</td>
<td>0.163</td>
<td>0.12</td>
<td>0.284</td>
<td>0.06</td>
<td>0.142</td>
</tr>
<tr>
<td>0.3</td>
<td>0.745</td>
<td>0.255</td>
<td>0.204</td>
<td>0.459</td>
<td>0.102</td>
<td>0.23</td>
</tr>
<tr>
<td>0.4</td>
<td>0.642</td>
<td>0.358</td>
<td>0.311</td>
<td>0.669</td>
<td>0.155</td>
<td>0.334</td>
</tr>
<tr>
<td>0.5</td>
<td>0.514</td>
<td>0.486</td>
<td>0.436</td>
<td>0.922</td>
<td>0.218</td>
<td>0.461</td>
</tr>
<tr>
<td>0.6</td>
<td>0.083</td>
<td>0.917</td>
<td>0.129</td>
<td>1.045</td>
<td>0.064</td>
<td>0.523</td>
</tr>
<tr>
<td>0.7</td>
<td>0.468</td>
<td>0.532</td>
<td>1.453</td>
<td>1.986</td>
<td>0.727</td>
<td>0.993</td>
</tr>
<tr>
<td>0.8</td>
<td>0.291</td>
<td>0.709</td>
<td>2.258</td>
<td>2.967</td>
<td>1.129</td>
<td>1.483</td>
</tr>
<tr>
<td>0.9</td>
<td>0.147</td>
<td>0.853</td>
<td>4.951</td>
<td>5.805</td>
<td>2.476</td>
<td>2.902</td>
</tr>
</tbody>
</table>
Conclusion

The results obtained in the queueing model discussed above are justified by means of a hypothetical example shown in the above tabular form. When $p_1=p_2$ the length of queue $L_{q_1}$ for high priority is less than the length of the queue $L_{q_2}$ for the low priority, due to the fact that the service is interrupted for the low priority queue. Further note that the waiting time for the high priority customer is less than the waiting time for the low priority customers.

References


6. Liping Wang(2012),”M/G/1 queues with vacation and priorities”


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