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Existence Conditions of Formal First Integrals of Lorenz System[1]

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Abstract

In the present paper, we study the formal first integrals of the well-known Lorenz system. First, for the simple condition of $s = 0$ we consider the form of the first integral. Furthermore, when $s \neq 0$ and b is not a negative rational number, the Lorenz system dose not have any nontrivial formal first integral in a neighborhood of the origin.

Keywords: First integral, Formal first integral, Lorenz system

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1 Introduction

The well-known Lorenz system is given by:

$$\begin{cases} \dot{x}_1 = s(x_2 - x_1), \\ \dot{x}_2 = rx_1 - x_2 - x_1 x_3, \\ \dot{x}_3 = -bx_3 + x_1 x_2 \end{cases} \tag{1}$$

where x_1, x_2, x_3 are real variables and s, r, b are real parameters. In this work, we will use a new method to prove the necessary condition of existing formal first integral of this system given by Jaume Llibre, see [4].

Consider the following autonomous different system

$$\dot{z} = f(z), \qquad z = (z_1, \cdots, z_n) \in C^n \tag{2}$$

where $f(z) = (f_1(z), \cdots, f_n(z))$ is a vector-valued function of dimension n.

Definition 1. A single-valued function $\Phi(z)$ is called a first integral of the system (2) if it is constant along any solution curve of the system (2). If $\Phi(z)$ is differentiable, it can be written as the condition

$$< \frac{\partial \Phi}{\partial z}, f(z) >= \sum_{i=1}^{n} \frac{\partial \Phi}{\partial z_i} f_i(z) \equiv 0 \cdot$$

If $\Phi(z)$ is a formal series in z, it is called a formal first integral of the system (2).

The following result due to Poincaré [5] presented a simple and easily verifiable criterion of nonexistence of formal first integrals for the system (2) .

Theorem 1. Assume that $f(z)$ is analytic and $f(0) = 0$. Let A be the Jacobi matrix of the vector field $f(z)$ at $z = 0$. If A is diagonalizable and its eigenvalues $\lambda_1, \cdots, \lambda_n$ do not satisfy any resonant equality of the following type

$$\sum_{j=1}^{n} k_j \lambda_j = 0, \quad k_j \in Z^+, \quad \sum_{j=1}^{n} k_j \geq 1. \tag{3}$$

Then the system (2) does not have any nontrivial formal first integral.

Generally, if the system (2) has a sufficiently rich set of first integrals such that its solutions can be expressed by these integrals, it is said to be completely integrable. If the system (2) does not admit any nontrivial first integrals, it is completely nonintegrable. Along the idea of Poincaré, a lot of works have been expanded, see [1, 2, 3, 4, 6].

2 Main Results

2.1 the case of $s = 0$

In particular, if s = 0, the Lorenz system must satisfy the resonant equality (3), then Theorem 1 is invalid. There are many works considering the cases that the systems are simply resonant, see [2, 3]. Here, we will consider the form of the first integrals for the system (1) under this simple resonance condition.

Theorem 2. Assume that $s=0$ and b is not a negative rational number. If $\Psi=\Psi(x_1,x_2,x_3)$ is a formal first integral of the Lorenz system (1), it is a formal power series with respect to the variables x_1.

Proof We rewrite the system as

$$\begin{cases} \dot{y}_1 = 0, \\ \dot{y}_2 = -y_2 - y_1 y_3, \\ \dot{y}_3 = -by_3 + y_1 y_2 + r y_1^2. \end{cases} \tag{4}$$

Let $\Phi=\sum_{k\geq 0}\Phi_k(y_2,y_3)y_1^k$ be a formal first integral of the system (4). By comparing the coefficients of the variable y_1^k,

$$<\frac{\partial\Phi_k(y_2,y_3)}{\partial y_2}y_1^k,-y_2>+<\frac{\partial\Phi_k(y_2,y_3)}{\partial y_3}y_1^k,-by_3>\equiv 0,$$

we have $\Phi_k(y_2,y_3)=C$ for all $k\geq 0$, i.e. the formal first integral of the system (4) is just dependent on the variable y_1. We also obtain that the formal first integral Ψ is just dependent on the variable x_1.

2.2 the case of $s\neq 0$

Since s is a parameter of the system, we treat the system (1) as the following system with respect to four variables x_1,x_2,x_3,s:

$$\begin{cases} \dot{x}_1 = s(x_2 - x_1), \\ \dot{x}_2 = r x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 = -b x_3 + x_1 x_2, \\ \dot{s} = 0. \end{cases} \tag{5}$$

Theorem 3. Suppose that $s\neq 0$ and b is not a negative rational number. If the system (5) has a formal first integral Ψ, then it must be only dependent on s.

Proof Transform the system (5) into

$$\begin{cases} \dot{u} = Au + f(u,v), \\ \dot{v}_1 = r v_1^2 - u_1 v_1, \\ \dot{v}_2 = 0, \end{cases} \tag{6}$$

where $u=(u_1.u_2)$, $v=(v_1,v_2)$, $A=diag(-1,-b)$, $f=(f_1,f_2)$ and

$$\begin{cases} f_1(u,v) = (r-1)v_1 v_2 - u_1 v_2, \\ f_2(u,v) = r(r-1)v_1 v_2 - r u_1 v_2 + u_2 v_1. \end{cases}$$

Let $v_1=\varepsilon\omega_1$, $v_2=\varepsilon\omega_2$, where ε is a positive constant and $\omega=(\omega_1,\omega_2)$, then the system (6) becomes

$$\begin{cases} \dot{u} = Au + f(u, \varepsilon\omega), \\ \dot{\omega}_1 = \varepsilon r \omega_1^2 - u_1 \omega_1, \\ \dot{\omega}_2 = 0. \end{cases} \tag{7}$$

So, we can write the formal first integral of the system (7) as

$$\overline{\Phi} = \varepsilon^{k_1+k_2} \sum_{k_1,k_2 \ge 0} \overline{\Phi}_{k_1,k_2}(u)\omega_1^{k_1}\omega^{k_2},$$

and it satisfies the following equation

$$< \frac{\partial \overline{\Phi}(u, \varepsilon\omega_1, \varepsilon\omega_2)}{\partial u}, Au + f(u, \varepsilon\omega_1, \varepsilon\omega_2) > + < \frac{\partial \overline{\Phi}(u, \varepsilon\omega_1, \varepsilon\omega_2)}{\partial \omega_1}, \varepsilon r \omega_1^2 - u_1 w_1 >$$

$$+ < \frac{\partial \overline{\Phi}(x, \varepsilon\omega_1, \varepsilon\omega_2)}{\partial \omega_2}, 0 > \equiv 0.$$

Similarly to the proof of Theorem 2, we compare the coefficients of the variable ε^m, $m = 0,1,\cdots$. By equating the coefficients of the variable ε^m,

$$< \varepsilon^m \frac{\partial \overline{\Phi}_{k_1,k_2}(u)}{\partial u_1} \omega_1^{k_1}\omega_2^{k_2}, -u_1 > + < \varepsilon^m \frac{\partial \overline{\Phi}_{k_1,k_2}(u)}{\partial u_2} \omega_1^{k_1}\omega_2^{k_2}, -bu_2 > + < k_1 \varepsilon^m \overline{\Phi}_{k_1,k_2}(u)\omega_1^{k_1-1}\omega_2^{k_2}, -u_1\omega_1 > \equiv 0,$$

we get

$$\overline{\Phi}_{k_1,k_2}(u) = C,$$

for all $k_1, k_2 \ge 0$ and $k_1 + k_2 = m$, where if $k_1 > 0$, then $C = 0$, i.e. the formal first integrals of the system (7) are just dependent on the variable ω_2. Also, we obtain that the formal first integral $\overline{\Psi}$ is just dependent on the variable s.

From Theorem 3 we get the following result for the Lorenz system (1).

Corollary 1. Suppose that $s \neq 0$ and b is not a negative rational. Then the Lorenz system (1) has no formal first integral in a neighborhood of the origin.

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