Results on Generalized Mittag-Leffler Function via Laplace Transform

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Abstract
The present paper deals with the results involving Generalized Mittag-Leffler function by using Laplace Transform.

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1 Introduction

In 1903, Gosta Mittag-Leffler [3] introduced the function \( E_\alpha(z) \) as

\[
E_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (\alpha \in \mathbb{C}, \Re(\alpha) > 0) \tag{1}
\]

In 1905, Wiman [7] studied the generalization of \( E_\alpha(z) \) as

\[
E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (\alpha \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0) \tag{2}
\]

In 1971, Prabhakar [4] introduced the function \( E_{\alpha,\beta}^\gamma(z) \) defined as,

\[
E_{\alpha,\beta}^\gamma(z) = \sum_{n=1}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \cdot \frac{z^n}{n!} \quad (\alpha \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0) \tag{3}
\]

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where \((\gamma)_n\) is the Pochhammer symbol defined as,

\[
(\gamma)_n = \begin{cases} 
1 & (n = 0; \gamma \neq 0) \\
\gamma(\gamma + 1) \ldots (\gamma + n - 1) & (n \in \mathbb{N}; \gamma \in \mathbb{C})
\end{cases}
\]

where \(\mathbb{N}\) being the set of positive integers.

If \(\gamma\) is neither zero nor a negative integer, then

\[
(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)}
\]

Shukla and Prajapati [6] introduced the function \(E_{\alpha,\beta}^{\gamma,q}(z)\), defined by

\[
E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=1}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \cdot \frac{z^n}{n!},
\]

where \(\alpha \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, q \in (0, 1) \cup \mathbb{N}\). (4)

The function \(E_{\alpha,\beta}^{\gamma,q}(z)\) converges absolutely for all \(z \in \mathbb{C}\) if \(q < \Re(\alpha) + 1\). Shukla and Prajapati [5] used the following result,

\[
(1 - z)^{-\gamma,q} = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{n!} z^n.
\]

(5)

**Laplace Transform**

The Laplace transform \([2]\) of a real scalar function \(f(x)\) of the real variable \(x\), with parameter \(t\), is defined as

\[
L_f(t) = \int_0^{\infty} e^{-tx} f(x) dx, \quad \text{whenever } L_f(t) \text{ exists.} \quad (6)
\]

**2 Main Result**

**Theorem.**

If \(\alpha, \beta, \gamma, \mu \in \mathbb{C} (\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\mu) > 0)\) and \(q \in (0, 1) \cup \mathbb{N}\), then,

\[
L \left[ (x - a)^{\beta+\mu-1} \Gamma(\mu) E_{\alpha,\beta+\mu}^{\gamma,q}(x - a)^{\alpha} \right] = \Gamma(\mu) e^{-\alpha t} t^{-\beta-\mu} (1 - t^{-\alpha})^{-\gamma,q}
\]

(7)

where, \(|t^{-\alpha}| < 1, L_f(t) \text{ is defined as } (6), E_{\alpha,\beta+\mu}^{\gamma,q}(x - a)^{\alpha} \text{ is defined as } (4)\) and \((1 - t^{-\alpha})^{-\gamma,q} \text{ is defined as } (5)\).
Proof.

\[ \text{L.H.S.} = L[(x-a)^{\beta+\mu-1}\Gamma(\mu)E_{\alpha,\beta+\mu}(x-a)^{\alpha}] \]

\[ = \frac{\Gamma(\mu)}{e^{at}} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\beta + \mu + \alpha n)} \frac{1}{n!} \frac{\Gamma(\beta + \mu + \alpha n)}{t^{\beta+\mu+\alpha n}} \]

\[ = \Gamma(\mu)e^{-at}t^{-\beta-\mu}(1-t^{-\alpha})^{-\gamma,q}, |t^{-\alpha}| < 1 \]

\[ = \text{R.H.S.} \]

Hence,

\[ L[(x-a)^{\beta+\mu-1}\Gamma(\mu)E_{\alpha,\beta+\mu}(x-a)^{\alpha}] = \Gamma(\mu)e^{-at}t^{-\beta-\mu}(1-t^{-\alpha})^{-\gamma,q} \]

\[ \square \]

2.1 Corollaries

Corollary I

For \( \alpha, \beta, \gamma, \delta \in \mathbb{C} (\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta) > 0) \)

\[ L[(x-a)^{\gamma-\alpha-1}E_{\beta,\gamma-\alpha}(\alpha(x)^{\alpha})] = t^{\gamma-\alpha}(1-at^{-\beta})^{-\delta} \] (8)

where, \(|t^{-\beta}| < 1, L_f(t)\) is defined as (6) and \(E_{\beta,\gamma-\alpha}(ax^\beta)\) is defined as (3).

Corollary II

For \( \alpha, \beta \in \mathbb{C} (\Re(\alpha), \Re(\beta) > 0) \)

\[ L \left[ \frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} + ax^{\beta-1}E_{\alpha,\beta}(ax^{\alpha}) \right] = t^{\beta} \left[ t^{\alpha} + a(1-at^{-\alpha})^{-1} \right] \] (9)

where, \(|t^{-\alpha}| < 1, L_f(t)\) is defined as (6) and \(E_{\alpha,\beta}(ax^{\alpha})\) is defined as (2).

Corollary III

For \( x > a \alpha, \lambda \in \mathbb{C} (\Re(\alpha) > 0) \)

\[ L \left[ e^{\lambda a}(x-a)^{\alpha}E_{1,\alpha+1}(\lambda x - \lambda a) \right] = e^{(\lambda-t a)(\alpha+1)} \left( 1 - \frac{\lambda}{t} \right)^{-1} \] (10)

where, \(|t^{-\alpha}| < 1, \) where \( L_f(t) \) is defined by equation (6) and \(E_{1,\alpha+1}(\lambda x - \lambda a)\) is given by equation (2).
2.2 Some Special Cases

Case I:

If \( \alpha > 0, \gamma > 0 \), then

\[
L \left[ x^{\alpha \gamma - 1} E^{\gamma}_{\alpha \gamma, \alpha}(-x^\alpha) \right] = (1 + t^\alpha)^{-\gamma}; |t^\alpha| < 1
\]

where \( L_f(t) \) is defined as (6) , \( E^{\gamma}_{\alpha \gamma, \alpha}(-x^\alpha) \) is defined as (3) and given by Mathai et al. [1].

Case II:

If \( \alpha > 0 \), then

\[
L \left[ x^{\alpha - 1} E^{\alpha}_{\alpha, \alpha}(-x^\alpha) \right] = (1 + t^\alpha)^{-1}; |t^\alpha| < 1
\]

where \( L_f(t) \) is defined as (6) , \( E^{\alpha}_{\alpha, \alpha}(-x^\alpha) \) is defined as (2) and given by Mathai et al. [1].

References


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