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Results on Generalized Mittag-Leffler Function via Laplace Transform

P. V. Shah [1] and R. K. Jana

Department of Applied Mathematics and Humanities
S. V. National Institute of Technology
Surat-395007 Gujarat, India
pratikshah8284@yahoo.co.in
rkjana2003@yahoo.com

Abstract

The present paper deals with the results involving Generalized Mittag-Leffler function by using Laplace Transform.

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1 Introduction

In 1903,Gosta Mittag-Leffler [3] introduced the function $E_\alpha(z)$ as

$$E_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad (\alpha \in \mathbf{C}, \Re(\alpha) > 0) \tag{1}$$

In 1905, Wiman [7] studied the generalization of $E_\alpha(z)$ as

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad (\alpha \in \mathbf{C}, \Re(\alpha) > 0, \Re(\beta) > 0) \tag{2}$$

In 1971, Prabhakar [4] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ defined as,

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \cdot \frac{z^n}{n!}$$
$$(\alpha \in \mathbf{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0) \tag{3}$$

[1]Permanent address: C. K. Pithawalla College of Engg. and Technology, Surat - 395 007, Gujarat, India

where $(\gamma)_n$ is the Pochmmer symbol defined as,

$$(\gamma)_n = \begin{cases} 1 & (n = 0; \gamma \neq 0) \\ \gamma(\gamma+1)\dots(\gamma+n-1) & (n \in \mathbf{N}; \gamma \in \mathbf{C}) \end{cases}$$

where, $\mathbf{N}$ being the set of positive integers.
If γ is neither zero nor a negative integer, then

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$$

Shukla and Prajapati [6] introduced the function$E^{\gamma,q}_{\alpha,\beta}(z)$,defined by

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \cdot \frac{z^n}{n!},$$

$$\text{where}, \alpha \in \mathbf{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, q \in (0,1) \cup \mathbf{N}. \tag{4}$$

The function $E^{\gamma,q}_{\alpha,\beta}(z)$ converges absolutely for all $z \in \mathbf{C}$ if $q < \Re(\alpha)+1$. Shukla and Prajapati [5] used the following result,

$$(1-z)^{-\gamma,q} = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{n!} z^n. \tag{5}$$

Laplace Transform

The Laplace transform [2] of a real scalar function $f(x)$ of the real variable x, wtih parameter t, is defined as

$$L_f(t) = \int_0^{\infty} e^{-tx} f(x)dx, \qquad \text{whenever } L_f(t) \text{exists.} \tag{6}$$

2 Main Result

Theorem.

If $\alpha, \beta, \gamma, \mu \in \mathbf{C}\, (\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\mu) > 0)$ *and* $q \in (0,1) \cup \mathbf{N}$.then,

$$L\left[(x-a)^{\beta+\mu-1}\Gamma(\mu)E^{\gamma,q}_{\alpha,\beta+\mu}(x-a)^{\alpha}\right] = \Gamma(\mu)e^{-\alpha t}t^{-\beta-\mu}(1-t^{-\alpha})^{-\gamma,q} \tag{7}$$

where, $|t^{-\alpha}| < 1, L_f(t)$ *is defined as* (6), $E^{\gamma,q}_{\alpha,\beta+\mu}(x-a)^{\alpha}$ *is defined as* (4)*and* $(1-t^{-\alpha})^{-\gamma,q}$ *is defined as*(5).

Proof.

$$\text{L.H.S.} = L\left[(x-a)^{\beta+\mu-1}\Gamma(\mu)E^{\gamma,q}_{\alpha,\beta+\mu}(x-a)^{\alpha}\right]$$

$$= \frac{\Gamma(\mu)}{e^{at}}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\beta+\mu+\alpha n)}\frac{1}{n!}\frac{\Gamma(\beta+\mu+\alpha n)}{t^{\beta+\mu+\alpha n}}$$

$$= \Gamma(\mu)e^{-at}t^{-\beta-\mu}(1-t^{-\alpha})^{-\gamma,q}, |t^{-\alpha}| < 1$$

$$= \text{R.H.S.}$$

Hence,

$$L\left[(x-a)^{\beta+\mu-1}\Gamma(\mu)E^{\gamma,q}_{\alpha,\beta+\mu}(x-a)^{\alpha}\right] = \Gamma(\mu)e^{-\alpha t}t^{-\beta-\mu}(1-t^{-\alpha})^{-\gamma,q}$$ □

2.1 Corollaries

Corollary I

For $\alpha, \beta, \gamma, \delta \in \mathbf{C}\,(\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta) > 0)$

$$L\left[(x-a)^{\gamma-\alpha-1}E^{\delta}_{\beta,\gamma-\alpha}(a(x)^{\alpha})\right] = t^{\gamma-\alpha}(1-at^{-\beta})^{-\delta} \tag{8}$$

where, $|t^{-\beta}| < 1$, $L_f(t)$ is defined as (6) and $E^{\delta}_{\beta,\gamma-\alpha}(ax^{\beta})$ is defined as (3).

Corollary II

For $\alpha, \beta \in \mathbf{C}\,(\Re(\alpha), \Re(\beta) > 0)$

$$L\left[\frac{x^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} + ax^{\beta-1}E_{\alpha,\beta}(ax^{\alpha})\right] = t^{-\beta}\left[t^{\alpha} + a(1-at^{-\alpha})^{-1}\right] \tag{9}$$

where, $|t^{-\alpha}| < 1$, $L_f(t)$ is defined as (6) and $E_{\alpha,\beta}(ax^{\alpha})$ is defined as (2).

Corollary III

For $x > a\ \alpha, \lambda \in \mathbf{C}\,(\Re(\alpha > 0)$

$$L\left[e^{\lambda a}(x-a)^{\alpha}E_{1,\alpha+1}(\lambda x - \lambda a)\right] = e^{(\lambda-t)a}t^{-(\alpha+1)}\left(1-\frac{\lambda}{t}\right)^{-1} \tag{10}$$

where,$|\frac{\lambda}{t}| < 1$, where $L_f(t)$ is defined by equation (6) and $E_{1,\alpha+1}(\lambda x - \lambda a)$ is given by equation (2).

2.2 Some Special Cases

Case I:

$$\text{If} \alpha > 0, \gamma > 0, then$$

$$L\left[x^{\alpha\gamma-1}E^{\gamma}_{\alpha\gamma,\alpha}(-x^{\alpha})\right] = (1+t^{\alpha})^{-\gamma}; |t^{\alpha}| < 1 \tag{11}$$

where $L_f(t)$ is defined as (6) , $E^{\gamma}_{\alpha\gamma,\alpha}(-x^{\alpha})$ is defined as (3) and given by Mathai et al. [1].

Case II:

$$\text{If} \alpha > 0, then$$

$$L\left[x^{\alpha-1}E_{\alpha,\alpha}(-x^{\alpha})\right] = (1+t^{\alpha})^{-1}; |t^{\alpha}| < 1 \tag{12}$$

where $L_f(t)$ is defined as (6) , $E_{\alpha,\alpha}(-x^{\alpha})$ is defined as (2) and given by Mathai et al. [1].

References

[1] Mathai, A.M., Saxena, R.K. and Haubold, H. J., The H-Function (Theory and Applications), Springer, New York, 2010.

[2] Miller, K.S. and Ross, B., An introduction to fractional calculus and fractional differential equations, Wiley, New York (1993).

[3] Mittag-Leffler, G.M., Sur la Nouvelle function , C.R. Acad. Sci. Paris No 137(1903), 554-558.

[4] Prabhakar, T.R., A singular integral equation with a generalized Mittag-Leffler function in the Kernel, Yokohama Math. J. No 19 (1971) 7-15.

[5] Prajapati J.C. and Shukla A.K., Decomposition of generalized Mittag-Leffler function and its properties, Advances in Pure Mathematics 2(2012), 8-14.

[6] Shukla, A. K. and Prajapati, J.C., On a generalization of Mittag-Leffler function and its properties, J. Math. Anal. Appl. No 336 (2007), 797-811.

[7] Wiman, A., Uber de fundamental satz in der theorie der funktionen . Acta Math. No 29 (1905), 191-201.

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