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Reduced-order Model Based on $\mathcal{H}_\infty$-Balancing for Infinite-Dimensional Systems

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Abstract

This paper presents a model reduction for unstable infinite dimensional system (A, B, C) using $\mathcal{H}_\infty$-balancing. To construct $\mathcal{H}_\infty$-balanced realization, we find a Lyapunov-balanced realization of a normalized left-coprime factorization (NLCF) of the scaled system $(A, \beta B, C)$. Next, we apply the new coordinate transformation to obtain yet another realization of NLCF system. This result is then translated to have the new scaled system $(A^t, \beta B^t, C^t)$ which similar with $(A, \beta B, C)$. Furthermore, it can be verified that the solutions of a control and filter $\mathcal{H}_\infty$-Riccati operator equations of the system (A^t, B^t, C^t) are equal and diagonal. This implies that the system (A^t, B^t, C^t) is $\mathcal{H}_\infty$-balanced realization of the system (A, B, C). Based on the small $\mathcal{H}_\infty$-characteristic values, the state variables of the system (A^t, B^t, C^t) is truncated, to yield a reduced-order model of the system (A, B, C). To demonstrate the effectiveness of the proposed method, numerical simulations are applied to Euler-Bernoulli beam equation.

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1 Introduction

Many of the dynamical systems are described by partial differential equation such as flexible beam, cable mass, heat distribution, wave equation and so on. State space formulation for such system requires infinite dimensionality. In practice, designing control for that system is impossible since it is of infinite state dimension. Therefore, it is very important to have a low order controller of infinite-dimensional systems.

In general, there are two approach to obtain low order controller of infinite-dimensional systems, namely *reduce-then-design* and *design-then-reduce* [1]. Model reduction techniques have become a very active research subject the last few decades. Various methods of model reduction of infinite-dimensional systems is much currently developing such as balanced truncation [2, 8], Hankel norm approximation [3] and reciprocal approach [6] has been successfully applied to infinite-dimensional systems. We term the techniques as *reduce-then-design* category.

The above description is the first alternative to obtain a controller of order is low. The weakness of this technique is not involve any information about controller design of the original system. This prompted the investigation of model reduction method that also involves the design strategy controller before the physical aspects of the system is lost when the process reduction. We call this approach as *design-then-reduce*. As already known, the Linear Quadratic Gaussian (LQG) or $\mathcal{H}_\infty$ controller design depends on the existence of solutions of two Riccati equations. This fact suggest a new of reduction techniques that the system is balanced around the solutions of the control and filter Riccati equations.

Curtain and Opmeer [4, 13] have developed model reduction method based on LQG-balanced realization for infinite dimensional system. The LQG-balanced is a realization of a transformed system, such that the solutions of the corresponding control and filter Riccati operator equations of LQG-type controllers are equal and diagonal. The existence of a LQG-balanced system is constructed via normalized left-coprime factorization, as is done in Meyer [10]. Hence, LQG-balanced truncation can be claimed of as a *design-then-reduce* approach.

During the last decades, there has been much research in the design of $\mathcal{H}_\infty$ controllers, which are robust to system uncertainty and disturbance. In [12], it is shown that the LQG-balanced realization of finite-dimensional linear time invariant (FDLTI) systems can be carried out to the $\mathcal{H}_\infty$-balanced based on $\mathcal{H}_\infty$-type controllers. The key step in the analysis of $\mathcal{H}_\infty$-balanced realization is to construct the existence of normalized left-coprime factorization (NLCF) systems using the solutions of Riccati equations of $\mathcal{H}_\infty$ controller. This idea give rise to generalize the $\mathcal{H}_\infty$-balanced realization to infinite dimensional systems. The systems considered are assumed to be exponentially stabilizable and detectable linear state, with bounded and finite-rank input and output operators. The purpose of this paper is to extended a model reduction based on $\mathcal{H}_\infty$-balancing via NLCF of infinite-dimensional systems.

2 $\mathcal{H}_\infty$-Control for Infinite-Dimensional Systems

In this section, we summarize without proof the existence of $\mathcal{H}_\infty$-control for infinite-dimensional systems. For the proofs we refer the reader to [11]. The infinite dimensional systems can be described in abstract form as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where A is an infinitesimal generator of C_0-semigroup $S(t)$ on Hilbert space $\mathcal{X}$, B is a bounded linear operator from a Hilbert space $\mathcal{U}$ to $\mathcal{X}$, C is a bounded linear operator from $\mathcal{X}$ to a Hilbert space $\mathcal{Y}$, and D is a bounded operator from $\mathcal{U}$ to $\mathcal{Y}$. We assume that the input operator B and the output operator C are of finite rank. Let $\mathbf{C}$ be the set of complex numbers. Choose $\mathcal{U} = \mathbf{C}^m$ and $\mathcal{Y} = \mathbf{C}^k$. The signal $u(t) \in \mathcal{L}_2([0,\infty);\mathbf{C}^m)$ is control input and $y(t) \in \mathcal{L}_2([0,\infty);\mathbf{C}^k)$ is the output.

We shall denote the state linear system given by (1) as (A,B,C,D) and transfer function given by $\mathbf{G}$, with realization $\mathbf{G}(s) = C(sI-A)^{-1}B + D$. We will omit the operator "D" if it is not relevant. The adjoint of operator A is written as A^* and domain of A is denoted by $\mathbf{D}(A)$. A symmetric operator A is *self-adjoint* if $\mathbf{D}(A^*) = \mathbf{D}(A)$. The self-adjoint operator A on the Hilbert space $\mathcal{X}$ with its inner product $\langle\cdot,\cdot\rangle$ is *nonnegative* if $\langle Az, z\rangle \geq 0$ for all $z \in \mathbf{D}(A)$ and *positive* if $\langle Az, z\rangle > 0$ for all nonzero $z \in \mathbf{D}(A)$. We shall use the notation $A \geq 0$ for nonnegativity of the self-adjoint operator A, and $A > 0$ for positivity.

In this study, we will assume the system (A,B,C) to be exponentially stabilizable and detectable. The system (A,B,C) is *exponentially stabilizable* if there exists an operator F such that $A+BF$ is exponentially stable and it is *exponentially detectable* if there exists an operator L such that $A+LC$ is exponentially stable. The operators $A+BF$ and $A+LC$ are exponentially stable if they generate the exponentially stable C_0-semigroup $S_F(t)$ and $S_L(t)$, respectively, on $\mathcal{X}$. Recall that the C_0-semigroup $S_F(t)$ on $\mathcal{X}$ is exponentially stable [5] if there exist positive constants M and α such that $\|S_F(t)\| \leq Me^{-\alpha t}$, for all $t \geq 0$.

Next, we review an $\mathcal{H}_\infty$-control problem for infinite-dimensional systems.

Let $\mathbf{P}$ be the transfer function of a generalized plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw_1(t) + Bu(t) \\ z_1(t) &= Cx(t), \quad z_2(t) = u(t) \\ y(t) &= Cx(t) + w_2(t). \end{aligned} \tag{2}$$

Put $z := [z_1 \quad z_2]'$ and $w := [w_1 \quad w_2]'$ where z is the error signals and w is the disturbance signals. The state space description of the controller with transfer function $\mathbf{K}$ is given by

$$\begin{aligned} \dot{x}_K(t) &= A_K x_K(t) + B_K y(t) \\ u(t) &= C_K x_K(t). \end{aligned} \tag{3}$$

The closed-loop transfer function from w to z will be denoted by $\mathbf{T}_{zw}$. The existence of suboptimal $\mathcal{H}_\infty$-control for infinite-dimensional systems is given by the following theorem.

Theorem 2.1 *[11] There exists an admissible controller* $\mathbf{K}$ *for the plant* $\mathbf{P}$ *such that* $\|\mathbf{T}_{zw}\|_\infty < \gamma$, $\gamma > 0$ *if and only if the following three conditions are satisfied*

1. *There exists* $X \in \mathrm{L}(\mathcal{X})$ *with* $X = X^* \geq 0$ *satisfying control Riccati equation*

$$A^*Xx + XAx - (1-\gamma^{-2})XBB^*Xx + C^*Cx = 0, \; x \in \mathbf{D}(A) \tag{4}$$

   such that $A_X = A - (1-\gamma^{-2})BB^*X$ *exponentially stable.*

2. *There exists* $Y \in \mathrm{L}(\mathcal{X})$ *with* $Y = Y^* \geq 0$ *satisfying filter Riccati equation*

$$AYx + YA^*x - (1-\gamma^{-2})YC^*CYx + BB^*x = 0, \; x \in \mathbf{D}(A^*) \tag{5}$$

   such that $A_Y = A - (1-\gamma^{-2})YC^*C$ *exponentially stable.*

3. $r_\sigma(YX) < \gamma^2$.

3 $\mathcal{H}_\infty$-Balanced Truncation

In this section, we will construct the $\mathcal{H}_\infty$-balanced realization for stabilizable and detectable infinite-dimensional systems (A, B, C) described by Eq. (1) with $D = 0$. Let $\mathbf{G}(s)$ be the transfer function of the system (A, B, C). The concept of the $\mathcal{H}_\infty$-balanced realization is given by the following definition.

Definition 3.1 *The system* $(\tilde{A}, \tilde{B}, \tilde{C})$ *is called an* $\mathcal{H}_\infty$-balanced realization *of* $\mathbf{G}$ *if there exist bounded, self-adjoint, nonnegative solutions* $\tilde{X}$ *and* $\tilde{Y}$ *of the control (4) and filter (5) Riccati equations, respectively, have identical solutions. Furthermore,* $\tilde{X} = \tilde{Y} = \Lambda$ *is a diagonal operator.*

The main tools for constructing $\mathcal{H}_\infty$-balancing is a connection between the exponentially stabilizable and detectable systems and the exponentially stable NLCF systems. The definition of NLCF of infinite-dimensional systems refers to Curtain and Zwart [5]. Next, we will construct the NLCF system based on the filter $\mathcal{H}_\infty$-Riccati equation (5). Let γ_0 denote the smallest γ for which admissible controller $\mathbf{K}$ exists. Define

$$\beta := \sqrt{1-\gamma^{-2}}, \tag{6}$$

where $\gamma > \max\{1, \gamma_0\}$, such that $0 < \beta \leq 1$. Multiply (5) with β^2 such that we have

$$A(\beta^2 Y)x + (\beta^2 Y)A^* x - \beta^4 Y C^* C Y x + \beta^2 B B^* x = 0, \tag{7}$$

$x \in \mathbf{D}(A^*)$. It is shown that the filter Riccati equation (7) is the filter LQG-Riccati equation [4] for the scaled system $(A, \beta B, C)$. Meanwhile, the control LQG-Riccati equation for $(A, \beta B, C)$ is equal to the control $\mathcal{H}_\infty$-Riccati equation (4), which can be rewritten as follows

$$A^* X x + X A x - X \beta^2 B B^* X x + C^* C x = 0,\ x \in \mathbf{D}(A). \tag{8}$$

Hence, $\beta^2 Y$ is a solution of the filter LQG-Riccati equation that stabilize the scaled system $(A, \beta B, C)$ due to $A_Y = A - \beta^2 Y C^* C$ is exponentially stable.

Let $\beta\mathbf{G}(s) = C(sI - A)^{-1}\beta B$ denote the transfer function of $(A, \beta B, C)$. Using the equation (7), we can construct NLCF from the scaled system $(A, \beta B, C)$. The following Lemma shows that (A_Y, B_Y, C_Y, D_Y) is a state-space realization of NLCF of $\beta\mathbf{G}$, with

$$A_Y = A - \beta^2 Y C^* C, \quad B_Y = [\beta B \quad -\beta^2 Y C^*],\ C_Y = C, \quad D_Y = [0 \quad I]. \tag{9}$$

Lemma 3.2 *[7] (A_Y, B_Y, C_Y, D_Y) given by (9) is a state-space realization of NLCF for $\beta\mathbf{G}$ with the transfer function is $[\tilde{\mathbf{N}}(s) \quad \tilde{\mathbf{M}}(s)]$, where $\tilde{\mathbf{N}}(s)$ and $\tilde{\mathbf{M}}(s)$ are given by*

$$\tilde{\mathbf{N}}(s) = C(sI - A_Y)^{-1}\beta B,\ \tilde{\mathbf{M}}(s) = I - C(sI - A_Y)^{-1}\beta^2 Y C^*.$$

such that

1. *$\beta\mathbf{G}(s) = \tilde{\mathbf{M}}(s)^{-1}\tilde{\mathbf{N}}(s)$.*

2. *There exist $\mathbf{X}$, and $\mathbf{Y}$, specifically $\mathbf{X}(s) = I + C(sI - A_X)^{-1}\beta^2 Y C^*$ and $\mathbf{Y}(s) = -\beta B^* X(sI - A_X)^{-1}\beta^2 Y C^*$, with $A_X = A - \beta^2 B B^* X$ such that satisfy*

$$\tilde{\mathbf{M}}(s)\mathbf{X}(s) - \tilde{\mathbf{N}}(s)\mathbf{Y}(s) = I, \quad s \in \mathbf{C}_0^+.$$

3. The normalization property is

$$\tilde{\mathbf{M}}(i\omega)\tilde{\mathbf{M}}(i\omega)^* + \tilde{\mathbf{N}}(i\omega)\tilde{\mathbf{N}}(i\omega)^* = I, \quad \omega \in \mathbf{R}.$$

Note that the NLCF system (A_Y, B_Y, C_Y, D_Y) is exponentially stable. The connection between the controllability and observability gramians of NLCF system with the solutions of the Riccati equations is given by the following Lemma.

Lemma 3.3 *[7] The controllability and observability L_B and L_C, respectively, of the NLCF system (A_Y, B_Y, C_Y, D_Y) satisfy*

$$L_B = \beta^2 Y, \quad L_C = X(I + \beta^2 YX)^{-1}, \; X = (I - L_C L_B)^{-1} L_C, \tag{10}$$

where $\beta^2 Y = \beta^2 Y^* \geq 0$ and $X = X^* \geq 0$ are the stabilizing solutions of the LQG-Riccati equations (7) and (8), respectively.

Since NLCF system (A_Y, B_Y, C_Y, D_Y) is exponentially stable, then Hankel singular values of $[\tilde{\mathbf{N}}(s) \quad \tilde{\mathbf{M}}(s)]$ is equal to

$$\sigma_i = \sqrt{\lambda_i(L_B L_C)}, \tag{11}$$

with $i = 1, 2, ...$ and satisfy $\sigma_1 < 1$ [5, Lemma 8.2.9, Lemma 9.4.7]. Furthermore, if the system NLCF is balanced realization, then

$$L_B = L_C = diag(\sigma_1, \sigma_2, ..., \sigma_r, ...), \qquad \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r \geq ... \geq 0.$$

According to the finite-dimensional terminology, we call the square roots of the eigenvalues of YX the $\mathcal{H}_\infty$-*characteristic values* of $\mathbf{G}$, i.e.,

$$\mu_i = \sqrt{\lambda_i(YX)}, \tag{12}$$

Since $r_\sigma(YX) < \gamma^2$, then $\mu_i < \gamma$ for $i = 1, 2, 3, ...$

Using the relation (10), (11) and (12), we can deduce the following simple relationship between the Hankel singular values of $[\tilde{\mathbf{N}}(s) \quad \tilde{\mathbf{M}}(s)]$ and the $\mathcal{H}_\infty$-characteristic values of $\mathbf{G}$ as follows

$$\sigma_i^2 = \lambda_i(L_B L_C) = \frac{\beta^2 \mu_i^2}{1 + \beta^2 \mu_i^2}. \tag{13}$$

From (13), we obtain

$$\mu_i = \frac{\sigma_i}{\beta\sqrt{1 - \sigma_i^2}}. \tag{14}$$

Notice that $1 > \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r \geq ... \geq 0$, so that for $\sigma_r > \sigma_{r+1}$ satisfy $1 - \sigma_r^2 < 1 - \sigma_{r+1}^2$. Hence, we deduce that $\frac{\sigma_r}{\beta\sqrt{1-\sigma_r^2}} > \frac{\sigma_{r+1}}{\beta\sqrt{1-\sigma_{r+1}^2}}$. As a result,

$\mathcal{H}_\infty$-characteristic values of $\mathbf{G}$ can be ordered as in the order of Hankel singular value of $[\tilde{\mathbf{N}}(s) \quad \tilde{\mathbf{M}}(s)]$ which satisfy

$$\gamma > \mu_1 \geq \mu_2 \geq ... \geq \mu_r \geq ... \geq 0. \tag{15}$$

The existence of $\mathcal{H}_\infty$-balancing for infinite dimensional systems is given by the following theorem. The basic idea to construct a realization of $\mathcal{H}_\infty$-balancing is modifying a method corresponding to the LQG-balancing [4].

Theorem 3.4 *Let* $\mathbf{G}$ *be transfer function of the systems* (A, B, C) *which are exponentially stabilizable and detectable with bounded, finite rank inputs and outputs. Then there exist an* $\mathcal{H}_\infty$*-balanced realization on the state space* ℓ_2 *for* $\mathbf{G}$ *such that the Riccati equations (4), (5) both have the solution*

$$\Lambda = diag\left(\frac{\sigma_1}{\beta\sqrt{1-\sigma_1^2}}, ...\right) = diag(\mu_1, ...).$$

Proof. Form the scaled system $(A, \beta B, C)$ of the system (A, B, C) corresponding to $\mathcal{H}_\infty$-Riccati filter (7). Because $\mathbf{G}(s) = C(sI - A)^{-1}B$, then the transfer function of $(A, \beta B, C)$ is $\beta\mathbf{G} = C(sI - A)^{-1}\beta B$. The next step, we construct the NLCF system (A_Y, B_Y, C_Y, D_Y) of the scaled system $(A, \beta B, C)$ with the operators A_Y, B_Y, C_Y, D_Y are given by (9). The transfer function of the NLCF system (A_Y, B_Y, C_Y, D_Y) is $[\tilde{\mathbf{N}}(s) \quad \tilde{\mathbf{M}}(s)]$, which satisfy $\beta\mathbf{G}(s) = \tilde{\mathbf{M}}(s)^{-1}\tilde{\mathbf{N}}(s)$.

Since the NLCF system (A_Y, B_Y, C_Y, D_Y) is exponentially stable, then the concept of Lyapunov-balanced realization can be applied to the NLCF system (A_Y, B_Y, C_Y, D_Y) [4]. Let $(\tilde{A}_Y, \tilde{B}_Y, \tilde{C}_Y, D_Y)$ be Lyapunov-balanced realization of the system (A_Y, B_Y, C_Y, D_Y) on the state-space ℓ_2. This is the system with controllability and observability gramians, $\tilde{L}_B, \tilde{L}_C$, both equal to $\tilde{L}_B = \tilde{L}_C = \Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_r, ...)$ and its transfer function is $[\tilde{\mathbf{N}}(s) \quad \tilde{\mathbf{M}}(s)]$. We now apply a new coordinate transformation

$$S = \beta^{\frac{1}{2}}\left(I - \Sigma^2\right)^{-\frac{1}{4}} : \ell_2 \to \ell_2, \tag{16}$$

to obtain yet another realization of $[\tilde{\mathbf{N}}(s) \quad \tilde{\mathbf{M}}(s)]$ on ℓ_2, the system $(A^\ell, B^\ell, C^\ell, D_Y)$ where

$$T^\ell(t) = S\tilde{T}_Y(t)S^{-1}, \quad A^\ell = S\tilde{A}_Y S^{-1}, \quad B^\ell = S\tilde{B}_Y, \quad C^\ell = \tilde{C}_Y S^{-1}, \tag{17}$$

and $\tilde{A}_Y$ is generator of C_0-semigroup $\tilde{T}_Y(t)$ on ℓ_2. Based on the realization (17) and transformation (16), immediately that the controllability gramian L_B^ℓ for the system $(A^\ell, B^\ell, C^\ell, D_Y)$ is ΣS^2 and its observabilty gramian L_C^ℓ is ΣS^{-2}.

Furthermore, we wish to identify that $(A^\ell, B^\ell, C^\ell, D_Y)$ is the NLCF system of the scaled system. Suppose system $(A^t, \beta B^t, C^t)$ is the candidate of the scaled system of $(A^\ell, B^\ell, C^\ell, D_Y)$. Let $\overline{Y} = \beta^2 Y$ be the solution of Riccati filter equation (7) for the system $(A^t, \beta B^t, C^t)$. Hence, the operators A^ℓ and B^ℓ can be expressed by

$$A^\ell = A^t - \overline{Y}(C^\ell)^* C^\ell, \quad B^\ell = \begin{bmatrix} B_1^\ell & -\overline{Y}(C^\ell)^* \end{bmatrix}. \tag{18}$$

Thus, the operators $A^t, \beta B^t$, and C^t are given by

$$A^t = A^\ell + \overline{Y}(C^\ell)^* C^\ell, \quad \beta B^t = B_1^\ell, \quad C^t = C^\ell. \tag{19}$$

The NLCF system $(A^\ell, B^\ell, C^\ell, D_Y)$ and the system $(A^t, \beta B^t, C^t)$ can be connected by the first term of (10) so obtained $L_B^\ell = \Sigma S^2 = \overline{Y}$. Then substitute (16) into ΣS^2 so we get

$$\overline{Y} = \Sigma S^2 = \beta^2 \Sigma \beta^{-1} \left(I - \Sigma^2\right)^{-\frac{1}{2}} = \beta^2 diag\left(\frac{\sigma_1}{\beta\sqrt{1-\sigma_1^2}}, ...\right) = \beta^2 \Lambda. \tag{20}$$

Next, we can verify that the system $(A^t, \beta B^t, C^t)$ has the transfer function $C^t(sI - A^t)^{-1}\beta B^t = \tilde{\mathbf{M}}(s)^{-1}\tilde{\mathbf{N}}(s) = \beta \mathbf{G}(s)$, with

$$\tilde{\mathbf{N}}(s) = C^t(sI - A^\ell)^{-1}\beta B^t, \quad \tilde{\mathbf{M}}(s) = I - C^t(sI - A^\ell)^{-1}\beta^2\Lambda(C^t)^*. \tag{21}$$

Using $\overline{Y} = \beta^2\Lambda$, Riccati filter equation (7) for the system $(A^t, \beta B^t, C^t)$ is given by

$$A^t\beta^2\Lambda x + \beta^2\Lambda(A^t)^* x - \beta^4\Lambda(C^t)^* C^t \Lambda x + \beta^2 B^t(B^t)^* x = 0, \tag{22}$$

for $x \in \mathbf{D}((A^t)^*)$. Thus, multiply (22) with β^{-2} so that we have

$$A^t\Lambda x + \Lambda(A^t)^* x - \beta^2\Lambda(C^t)^* C^t\Lambda x + B^t(B^t)^* x = 0, \tag{23}$$

$x \in \mathbf{D}((A^t)^*)$. Note that (23) is the $\mathcal{H}_\infty$-Riccati filter equation of the system (A^t, B^t, C^t) and its solution is denoted by $\tilde{Y}$, then $\tilde{Y}$ be diagonal operator $\tilde{Y} = \Lambda$. Since $\beta\mathbf{G} = C^t(sI - A^t)^{-1}\beta B^t$, then we have $\mathbf{G} = C^t(sI - A^t)^{-1}B^t$. Next, suppose that $\tilde{X}$ is the solution of $\mathcal{H}_\infty$-Riccati control for the system (A^t, B^t, C^t). By using the relation as in the third part of (10) satisfy

$$\begin{aligned} \tilde{X} &= (I - L_C^\ell L_B^\ell)^{-1} L_C^\ell = (I - \Sigma S^{-2}\Sigma S^2)^{-1}\Sigma S^{-2} \\ &= (I - \Sigma^2)^{-\frac{1}{2}}\beta^{-1}\Sigma \\ &= diag\left(\frac{\sigma_1}{\beta\sqrt{1-\sigma_1^2}}, \frac{\sigma_2}{\beta\sqrt{1-\sigma_2^2}}, ...\right) = \Lambda. \end{aligned} \tag{24}$$

So it proved that the system (A^t, B^t, C^t) is $\mathcal{H}_\infty$-balanced realization of the transfer function $\mathbf{G}$. □

Based on Theorem 3.4, reduced-order model may be obtained by truncating the state variables in an $\mathcal{H}_\infty$-balanced realization appropriately with small $\mathcal{H}_\infty$-characteristic values. We will use the infinite matrix representation for operators on ℓ_2 with respect to the usual orthonormal basis.

Suppose NLCF system $(A^\ell, B^\ell, C^\ell, [0 \quad I])$ has the controllability gramian $L_B^\ell = \beta\Sigma\,(I-\Sigma^2)^{-\frac{1}{2}}$ and the observability gramian $L_C^\ell = \Sigma\beta^{-1}\,(I-\Sigma^2)^{\frac{1}{2}}$. Thus, the scaled system $(A^t, \beta B^t, C^t)$ has control and filter LQG-Riccati equation solutions, $\tilde{X} = (I-\Sigma^2)^{-\frac{1}{2}}\beta^{-1}\Sigma$ and $\overline{Y} = \beta\Sigma\,(I-\Sigma^2)^{-\frac{1}{2}}$, respectively. Moreover, the control and filter $\mathcal{H}_\infty$-Riccati equation solutions corresponding to $\mathcal{H}_\infty$-balanced realization (A^t, B^t, C^t) are given by $\tilde{X} = \tilde{Y} = \Lambda = (I-\Sigma^2)^{-\frac{1}{2}}\beta^{-1}\Sigma$.

Choose a positive integer r such that $\mu_r > \mu_{r+1}$ and partitioned the $\mathcal{H}_\infty$-balanced system (A^t, B^t, C^t), scaled system $(A^t, \beta B^t, C^t)$ and NLCF system $(A^\ell, B^\ell, C^\ell, [0 \quad I])$ associated with $\Lambda = diag(\Lambda_1, \Lambda_2)$, where
$\Lambda_1 = diag(\mu_1, \mu_2, ..., \mu_r)$, $\Lambda_2 = diag(\mu_{r+1}, \mu_{r+2}, ...)$ as follows:

$$A^t = \begin{bmatrix} A_{11}^t & A_{12}^t \\ A_{21}^t & A_{22}^t \end{bmatrix},\ B^t = \begin{bmatrix} B_1^t \\ B_2^t \end{bmatrix},\ C^t = \begin{bmatrix} C_1^t & C_2^t \end{bmatrix},$$

$$A^t = \begin{bmatrix} A_{11}^t & A_{12}^t \\ A_{21}^t & A_{22}^t \end{bmatrix},\ \beta B^t = \begin{bmatrix} \beta B_1^t \\ \beta B_2^t \end{bmatrix},\ C^t = \begin{bmatrix} C_1^t & C_2^t \end{bmatrix},$$

$$A^\ell = \begin{bmatrix} A_{11}^\ell & A_{12}^\ell \\ A_{21}^\ell & A_{22}^\ell \end{bmatrix},\ B^\ell = \begin{bmatrix} B_{11}^\ell & B_{12}^\ell \\ B_{21}^\ell & B_{22}^\ell \end{bmatrix}, C^\ell = \begin{bmatrix} C_1^\ell & C_2^\ell \end{bmatrix}.$$

Using the method of truncation to the state variables corresponding to the small $\mathcal{H}_\infty$-characteristic values, we obtain the rth-order $\mathcal{H}_\infty$-balanced truncation of (A^t, B^t, C^t) is given by the finite-dimensional system (A_{11}^t, B_1^t, C_1^t). Thus, the rth-order truncation of the scaled system $(A^t, \beta B^t, C^t)$ is $(A_{11}^t, \beta B_1^t, C_1^t)$. Meanwhile, the rth-order truncation of the NLCF system $(A^\ell, B^\ell, C^\ell, [0 \quad I_r])$ is $(A_{11}^\ell, [B_{11}^\ell \quad B_{12}^\ell], C_1^\ell, [0 \quad I])$.

Let $\mathbf{G}^r$ and $\beta\mathbf{G}^r$ be the transfer functions of the $\mathcal{H}_\infty$-balanced truncation (A_{11}^t, B_1^t, C_1^t) and reduced-scaled system $(A_{11}^t, \beta B_1^t, C_1^t)$, respectively. Hence, the reduced-scaled system $(A_{11}^t, \beta B_1^t, C_1^t)$ can be viewed as an intermediary to connect the system $\mathcal{H}_\infty$-balanced truncation (A_{11}^t, B_1^t, C_1^t) and reduced-NLCF system $(A_{11}^\ell, [B_{11}^\ell \quad B_{12}^\ell], C_1^\ell, [0 \quad I])$, as described in the following Theorem.

Theorem 3.5 *The reduced-NLCF system $(A_{11}^\ell, [B_{11}^\ell \quad B_{12}^\ell], C_1^\ell, [0 \quad I])$ is the normalized left-coprime factor system of the reduced-scaled system $(A_{11}^t, \beta B_1^t, C_1^t)$.*

Proof. We consider that the $\mathcal{H}_\infty$-Riccati equations of (A_{11}^t, B_1^t, C_1^t) have solutions $\Lambda_1 = diag(\mu_1, \mu_2, ..., \mu_r)$ which correspond to the control and filter LQG-Riccati equations of the reduced-scaled system $(A_{11}^t, \beta B_1^t, C_1^t)$. It is also associated with the observability and controllability gramians of the reduced-NLCF system $(A_{11}^{\ell}, [B_{11}^{\ell} \quad B_{12}^{\ell}], C_1^{\ell}, [0 \quad I])$ via (10). Based on the block structure, it is clear that

$$A_{11}^{\ell} = A_{11}^{t} - \beta^2 \Lambda_1 (C_1^t)^* (C_1^t), [B_{11}^{\ell} \quad B_{12}^{\ell}] = [\beta B_{11}^{t} \quad -\beta^2 \Lambda_1 (C_1^t)^*], \quad C_1^{\ell} = C_1^{t}.$$

The transfer function of the reduced-NLCF system $(A_{11}^{\ell}, [B_{11}^{\ell} \quad B_{12}^{\ell}], C_1^{\ell}, [0 \quad I])$ can be expressed as a pair $[\tilde{\mathbf{N}}^r(s) \quad \tilde{\mathbf{M}}^r(s)]$ with

$$\tilde{\mathbf{N}}^r(s) = C_1^t (sI - A_{11}^{\ell})^{-1} \beta B_1^t \quad \text{dan} \tilde{\mathbf{M}}^r(s) = I - C_1^t (sI - A_{11}^{\ell})^{-1} \beta^2 \Lambda_1 (C_1^t)^*.$$

Using Lemma 3.2, we can verify that the reduced-NLCF system $(A_{11}^{\ell}, [B_{11}^{\ell} \quad B_{12}^{\ell}], C_1^{\ell}, [0 \quad I])$ is the normalized left-coprime factor system of the system $(A_{11}^t, \beta B_1^t, C_1^t)$ and $\beta \mathbf{G}^r(s) = \tilde{\mathbf{M}}^r(s)^{-1} \tilde{\mathbf{N}}^r(s)$, with $\beta \mathbf{G}^r$ is transfer function of $(A_{11}^t, \beta B_1^t, C_1^t)$. □

4 Numerical Computation

The fundamental idea in the $\mathcal{H}_\infty$-balanced truncation technique is that the system must be balanced based on the operator solutions of the control and filter $\mathcal{H}_\infty$-Riccati equations. In general, the operator solutions to the $\mathcal{H}_\infty$-Riccati equations cannot be sought analytically. Therefore, a convergent numerical schemes must be used as in [4]. This section presents the numerical computation for obtaining the model reduction procedure according to results described in Section 3. The algorithm of the $\mathcal{H}_\infty$-balanced truncation method can be summarized by the following steps.

1. Perform the approximating sequence (A^n, B^n, C^n) to (A, B, C), where n is the order of the approximation scheme.

2. Solve the filter $\mathcal{H}_\infty$-Riccati equation for (A^n, B^n, C^n) as given in (5).

3. Form the scaled system $(A^n, \beta B^n, C^n)$ where β is defined in (6).

4. Construct the NLCF system $(A_Y^n, B_Y^n, C_Y^n, D_Y^n)$ of the scaled system $(A^n, \beta B^n, C^n)$ as in (9).

5. Obtain the Lyapunov balanced realization $(\tilde{A}_Y^n, \tilde{B}_Y^n, \tilde{C}_Y^n, D_Y^n)$ of the NLCF system $(A_Y^n, B_Y^n, C_Y^n, D_Y^n)$.

6. Using the transformation (16), find the NLCF system $((A^n)^{\ell}, (B^n)^{\ell}, (C^n)^{\ell}, D_Y^n)$ as given by (17).

7. Compute the scaled system $((A^n)^t, \beta (B^n)^t, (C^n)^t)$ of the NLCF system $((A^n)^{\ell}, (B^n)^{\ell}, (C^n)^{\ell}, D_Y^n)$ as defined in (19).

8. Form the system $((A^n)^t, (B^n)^t, (C^n)^t)$ as $\mathcal{H}_\infty$-balanced realization of the system (A^n, B^n, C^n).

9. Based on the magnitude of the $\mathcal{H}_\infty$-characteristic values, determined a truncated system $((A^n)^t_{11}, (B^n)^t_1, (C^n)^t_1)$ of $((A^n)^t, (B^n)^t, (C^n)^t)$ as the rth-order $\mathcal{H}_\infty$-balanced truncation of (A^n, B^n, C^n).

5 Case study

In this section, the effectiveness of the proposed method approach is applied to the Euler-Bernoulli beam clamped at one end ($\xi = 0$) and free to vibrate at the other end ($\xi = \ell$). Here, $x(t, \xi)$ denotes the deflection of the beam at time t and position ξ. The deflection can be controlled by applying a torque $u(t)$ at the clamped end ($\xi = 0$). Using linear viscous damping (γ_1) and Kelvin-Voigt damping (γ_2), the partial differential equation of the beam is given by

$$\rho\frac{\partial^2 x}{\partial t^2}(t,\xi) + \gamma_1\frac{\partial x}{\partial t}(t,\xi) + EI_b\frac{\partial^4 x}{\partial \xi^4}(t,\xi) + \gamma_2 I_b\frac{\partial^5 x}{\partial t\partial \xi^4}(t,\xi) = \frac{\rho\xi}{I_h}u(t), \tag{25}$$

with ρ is density of the beam, E is the Young modulus, I_b is the moment of inertia, and I_h is the hub inertia. The boundary conditions are

$$x(t,0) = \frac{\partial x}{\partial \xi}(t,0) = 0,$$

$$EI_b\frac{\partial^2 x}{\partial \xi^2}(t,l) + \gamma_2 I_b\frac{\partial^3 x}{\partial t\partial \xi^2}(t,l) = 0,\ EI_b\frac{\partial^3 x}{\partial \xi^3}(t,l) + \gamma_2 I_b\frac{\partial^4 x}{\partial t\partial \xi^3}(t,l) = 0, \tag{26}$$

for $t > 0$ and $0 < \xi < l$. The values of the physical parameters in this example are taken as follows [9]:

parameter	value
E	2,1 $\times 10^{11}$ N/m^2
I_b	1,167 $\times 10^{-10}$ m^4
ρ	2,975 kg/m
γ_1	0,001 Ns/m^2
γ_2	0,01 Ns/m^2
I_h	121,9748 kgm^2

Let $\mathbf{H}$ be the closed linear subspace of the Sobolev space $\mathbf{H}^2(0,1)$

$$\mathbf{H} = \left\{ z \in \mathbf{H}^2(0,1) : z(0) = \frac{dz}{d\xi}(0) = 0 \right\}$$

and define the state-space to be $\mathcal{X} = \mathbf{H} \times \mathcal{L}_2(0,1)$ with state $x(t) = (x(t,\cdot), \frac{\partial x}{\partial t}(t,\cdot))$.

If the tip deflection of position is measured, a state-space formulation of the above partial differential equation can be presented in the following abstract form

$$A = \begin{bmatrix} 0 & I \\ -\frac{EI_b}{\rho}\frac{d^4}{d\xi^4} & -\frac{\gamma_2 I_b}{\rho}\frac{d^4}{d\xi^4} - \frac{\gamma_1}{\rho} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{\xi}{I_h} \end{bmatrix}, Cx(t) = \begin{bmatrix} x(t,l) \end{bmatrix},$$

with domain

$$\mathbf{D}(A) = \{(\phi,\psi) \in \mathcal{X} : \psi \in \mathbf{H} \text{ and } M = EI_b\phi_{\xi\xi} + \gamma_2 I_b \psi_{\xi\xi}, \\ M \in \mathbf{H}^2(0,1) \quad \text{with} \quad M(l) = M_\xi(l) = 0\}.$$

In [9], it is shown that operator A generates an exponentially stable semigroup on $\mathcal{X}$. If the operator A is exponentially stable analytic, then the system (A, B, C) is exponentially stabilizable and detectable [5, Lemma 7.3.2]. Hence, we can apply model reduction based on $\mathcal{H}_\infty$-balanced truncation method for this systems.

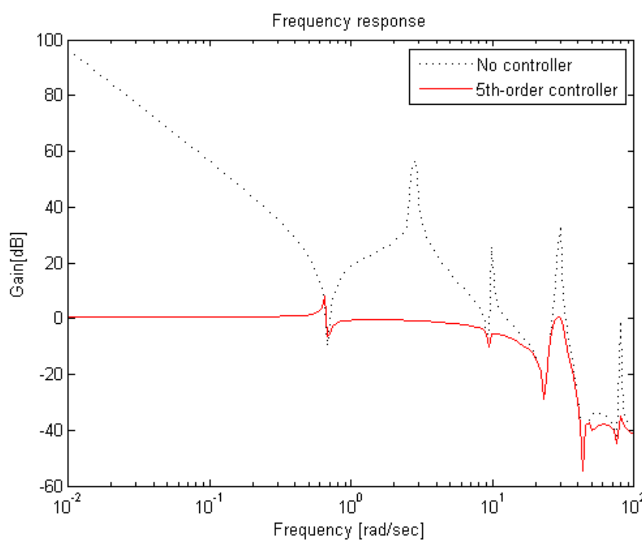

Figure 1: Frequency responses of open and closed-loop by 5th-order controller

To formulate a full order approximation, we use a Galerkin finite element method with cubic spline [14]. We compute the $\mathcal{H}_\infty$-balanced realization for 32 dimensional finite element method with the length of beam to be $l = 7$. The algorithm of the model reduction approaches discussed in Section 4 is applied to the systems in Eqs. (25) and (26).

Furthermore, we design a low order $\mathcal{H}_\infty$ controller based on the reduced-order model by $\mathcal{H}_\infty$-balanced truncation. The frequency responses of a open-loop dan closed-loop systems by 5th-order, and 10th-order controller of the 32

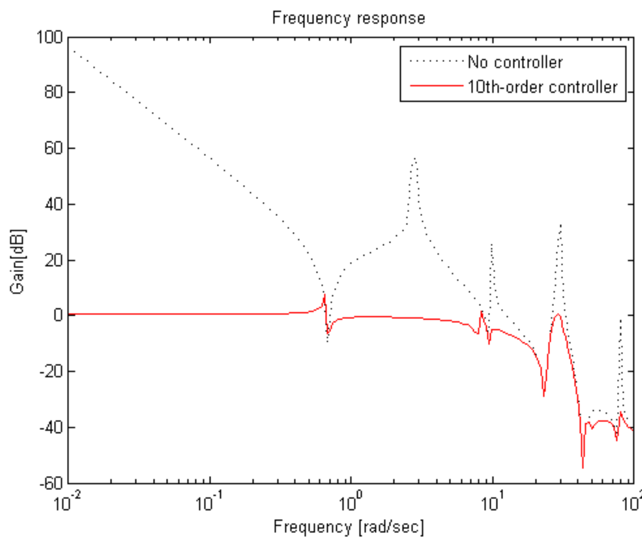

Figure 2: Frequency responses of open and closed-loop by 10th-order controller

dimensional finite-element method of the beam is shown in Fig. 1 and Fig. 2, respectively. It is verified that the low-order controller with the proposed method can minimize the peak of the open-loop system of the beam.

6 Conclusions

In this paper, we extended the model reduction method based on $\mathcal{H}_\infty$-balancing for infinite-dimensional system. The $\mathcal{H}_\infty$-balanced is a realization of a transformed system, such that the solutions of the corresponding Riccati control and Riccati filter operator equations are diagonal. The existence of a $\mathcal{H}_\infty$-balanced system is constructed via normalized left-coprime factorization. Based on the $\mathcal{H}_\infty$-balanced system, a balanced truncation of the state variables which correspond to small characteristics is done, to obtain a reduced model of finite dimension. From the simulations results, the proposed method can minimize the peak of the open-loop system of the beam.

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