

# Hankel Determinant Structure of the Rational Solutions for Fifth Painlevé Equation

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## Abstract

In this paper, we construct the Hankel determinant representation of the rational solutions for the fifth Painlevé equation through the Umemura polynomials. Our construction gives an explicit form of the Umemura polynomials  $\sigma_n$  for  $n \geq 0$  in terms of the Hankel Determinant formula. Besides, We compute the generating function of the entries in terms of logarithmic derivative of the Heun Confluent Function.

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## 1 Introduction

The six Painlevé transcendents are now regarded as the nonlinear version of the classical special functions [8]. It is known that the Painlevé transcendents cannot be expressed by the solution of the linear equations, except for two classes of solutions, namely, special functions solutions and the rational solutions. There are special polynomials associated with the rational solutions for the Painlevé equations. For instance,  $(P_{II})$  Yablonskii - Vorob'ev polynomials,  $(P_{IV})$  Okamoto polynomials and  $(P_{III}, P_V, P_{VI})$  Umemura polynomials [5, 6]. These special polynomials are defined by the second-order bilinear differential difference equations which are equivalent to Toda Equations.

Generally, the determinant structures are useful for understanding the nature of the polynomials mentioned above. For instance, it has been shown that the Okamoto and Umemura polynomials are nothing but a specialization of the 3-core Schur function and the Schur function respectively by using the

Jacobi-Trudi type determinant formula [3, 4, 5]. Recently, it is clarified that these determinant formulas can be also applied not only for the special (classical) solution or special polynomials associated with the Painlevé equation but also for generic transcendental ones [2, 7, 6].

In [1], they present the following theorem which gives the definition of the Hankel determinant:

**Theorem 1.1.** *Let  $\{a_n(t)\}_{n \in \mathbb{N}}$  and  $\{b_n(t)\}_{n \in \mathbb{N}}$  be the sequences defined recursively as*

$$a_n = a'_{n-1} + \psi \sum_{\substack{i+j=n-2 \\ i,j \geq 0}} a_i \cdot a_j, \quad b_n = b'_{n-1} + \varphi \sum_{\substack{i+j=n-2 \\ i,j \geq 0}} b_i \cdot b_j$$

where  $a_0 = \varphi$ ,  $b_0 = \psi$ . For any  $N \in \mathbb{Z}$ , the Hankel determinant  $\tau_N$  is given by

$$\tau_N = \begin{cases} \det(a_{i+j-2})_{i,j \leq N}, & N > 0, \\ 1, & N = 0, \\ \det(b_{i+j-2})_{i,j \leq |N|}, & N < 0. \end{cases} \tag{1.1}$$

Then  $\tau_N$  satisfies the Toda equation

$$\tau''_N \tau_N - (\tau'_N)^2 = \tau_{N+1} \tau_{N-1} - \phi \psi \tau_N^2 \tag{1.2}$$

where  $\tau_{-1} = \psi$ ,  $\tau_0 = 1$ ,  $\tau_1 = \varphi$ .

In this paper, we consider the well-known special polynomials associated with the rational solutions for the  $P_V$  which are called "Umemura polynomials". The Umemura polynomials  $\sigma_n(t, r)$  are polynomials generated by the recurrence relation [5, 9]:

$$t(\sigma''_n \sigma_n - (\sigma'_n)^2) + \sigma'_n \sigma_n + \left(\frac{t}{8} - r + \frac{3}{4}n\right) \sigma_n^2 = \sigma_{n+1} \sigma_{n-1} \tag{1.3}$$

with initial condition  $\sigma_0 = \sigma_1 = 1$ , where  $r \in \mathbb{C}$ . The rational function

$$y(t) = - \frac{\sigma_n(t, r + \frac{1}{2}) \sigma_{n+1}(t, r + \frac{1}{4})}{\sigma_n(t, r) \sigma_{n+1}(t, r + \frac{3}{4})}$$

solves the fifth painlevé equation  $P_V$ :

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}(\alpha y + \frac{\beta}{y}) + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1}$$

with the parameters

$$(\alpha, \beta, \gamma, \delta) = \left(2r^2, -2\left(r - \frac{n}{2}\right)^2, n, \frac{-1}{2}\right)$$

We present the following theorem which gives an explicit form of the Umemura polynomials  $\sigma_n = \sigma_n(t, r)$  for  $n \geq 0$  in terms of the Hankel Determinant formula.

**Theorem 1.2.** *The Umemura polynomials  $\sigma_n$  are given by the following Hankel determinant formula:*

$$\sigma_n = \begin{cases} 1, & n = 0, 1, \\ t^{-\frac{1}{2}n(n-1)} \det(a_{i+j-2})_{1 < i, j \leq n}, & n > 1. \end{cases} \tag{1.4}$$

where

$$\begin{aligned} a_n &= t(a'_{n-1} + \frac{3}{4}a_{n-1}) + t(\frac{t}{8} - r) \sum_{k=0}^{n-2} a_k a_{n-k-2} \\ a_0 &= 1, \quad a_1 = \frac{3}{4}t \end{aligned} \tag{1.5}$$

and also we discuss the generating function  $F(t, \lambda) = \sum_{n=0}^{\infty} a_n \lambda^{-n}$ , for the entries of our determinants and we find that  $F(t, \lambda)$  satisfies a partial differential equation with respect to  $t$  and it has a general solution which is given by the following theorem:

**Theorem 1.3.**  *$F(t, \lambda)$  can be given by*

$$F(t, \lambda) = \frac{\lambda}{(\frac{t}{8} - r)} \left[ \frac{\partial}{\partial t} \log Y_1 + \frac{1}{2t} (\lambda - \frac{3}{4}t) \right]$$

which is can recast in terms of confluent hypergeometric function when

$$Y_1 = \left(\frac{1}{4}\right)^{\frac{\lambda}{2} + \frac{3}{2}} t^{\frac{\lambda}{2} + 2} e^{-\frac{t}{8}} \left( C_1(\lambda) M(-\lambda, \lambda + 3, \frac{t}{4}) + C_2(\lambda) U(-\lambda, \lambda + 3, \frac{t}{4}) \right)$$

and Heun Confluent Function when

$$\begin{aligned} Y_1 &= C_1(\lambda) t^{1 + \frac{\lambda}{2}} e^{\frac{t}{8}} Hc(2r, 1 + \lambda, -2, r(3 + 3\lambda - 8r), \frac{1}{2}(-1 - 6r)\lambda + 8r^2 + \frac{1}{2}, \frac{t}{8r}) \\ &\quad + C_2(\lambda) t^{-\frac{\lambda}{2}} e^{\frac{t}{8}} Hc(2r, -1 - \lambda, -2, r(3 + 3\lambda - 8r), \frac{1}{2}(-1 - 6r)\lambda + 8r^2 + \frac{1}{2}, \frac{t}{8r}) \end{aligned}$$

where  $r = 0$  and  $r \neq 0$  respectively.

such that  $M$  and  $U$  are Kummer's functions of the first and second kind respectively and  $Hc$  is a Heun confluent function. In the next sections we will give our proofs of our results.

## 2 Proofs

In this section we give the proof of our theorem (1.2).

*Proof.* We present the following transformation

$$\sigma_n(t) = t^{-\frac{1}{2}n(n-1)}\rho_n(t) \quad (2.6)$$

and then we have from equation (1.3) that

$$t^2(\rho_n''\rho_n - (\rho_n')^2) + t\rho_n'\rho_n + t\left(\frac{t}{8} - r + \frac{3}{4}n\right)\rho_n^2 = \rho_{n+1}\rho_{n-1} \quad (2.7)$$

For  $n > 0$ , define the following transformation

$$\rho_n(t) = e^{-(\frac{3}{4}n)t}\tau_n(z) \quad (2.8)$$

where  $z = \log t$  and then from equation (2.7) we have

$$\frac{d^2\tau_n}{dz^2}\tau_n - \left(\frac{d\tau_n}{dz}\right)^2 = \tau_{n+1}\tau_{n-1} - \phi\psi\tau_n^2 \quad (2.9)$$

where

$$\psi = e^z e^{-\frac{3}{4}e^z} \left(\frac{e^z}{8} - r\right), \quad \tau_0 = 1, \quad \phi = e^{\frac{3}{4}e^z}$$

This is the Toda equation. By applying theorem 1.1 to equation (2.9) with  $n > 0$ , we have

$$\tau_n(z) = \det(c_{i+j-2}), \quad c_n = \frac{dc_{n-1}}{dz} + \psi \sum_{\substack{i+j=n-2 \\ i,j \geq 0}} c_i \cdot c_j, \quad c_0 = e^{\frac{3}{4}e^z} \quad (2.10)$$

Now, in order to construct the Hankel determinant formula for  $\sigma_n$  when  $n > 0$ , we have to find the Hankel determinant formula for  $\rho_n$  because  $\sigma_n = t^{-\frac{1}{2}n(n-1)}\rho_n$ .

Putting  $c_n = e^{\frac{3}{4}t}a_n$  in equation (2.10) and noticing that  $\rho_n(t) = e^{-\frac{3}{4}nt}\tau_n(z)$  with  $z = \log t$ , we obtain the formula for  $n > 0$  as in equation (1.5).  $\square$

### 2.1 Riccati Equation For Generating Functions

We consider the generating functions of the entries as the following formal series

$$F(t, \lambda) = \sum_{n=0}^{\infty} a_n \lambda^{-n} \quad (2.11)$$

where  $a_n$  are characterized by the recurrence relations (1.5). We obtain the following Riccati equation.

**Theorem 2.1.** *F satisfies the following partial differential equation*

$$t\lambda \frac{\partial F}{\partial t} = -t\left(\frac{t}{8} - r\right)F^2 + \left(\lambda^2 - \frac{3}{4}t\lambda\right)F - \lambda^2 \tag{2.12}$$

*Proof.* Equation (2.12) can be derived as follows :

$$\begin{aligned} t\left(\frac{t}{8} - r\right)F^2 &= t\left(\frac{t}{8} - r\right) \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k a_{n-k}\right)\lambda^{-n} \\ &= \sum_{n=0}^{\infty} (a_{n+2} - ta'_{n+1} - \frac{3}{4}ta_{n+1})\lambda^{-n} \\ &= (\lambda^2 F - a_o\lambda^2 - a_1\lambda) - t\left(\lambda \frac{\partial F}{\partial t} - \lambda a'_o\right) - \frac{3}{4}t(\lambda F - \lambda a_o) \\ &= -t\lambda \frac{\partial F}{\partial t} + \left(\lambda^2 - \frac{3}{4}t\lambda\right)F - \lambda^2 \end{aligned}$$

then equation (2.12) holds. □

### 3 Solution of the Riccati Equation

We linearize the Riccati equations to a system of two first order ODE. It is not difficult to derive the following theorem from theorem (2.1) :

**Theorem 3.1.** *It is possible to introduce the functions  $Y_1$  and  $Y_2$  consistently as:*

$$F(t, \lambda) = \frac{\lambda}{\left(\frac{t}{8} - r\right)} \left[ \frac{\partial}{\partial t} \log Y_1 + \frac{1}{2t} \left(\lambda - \frac{3}{4}t\right) \right] \tag{3.13}$$

and

$$Y_2 = \frac{1}{\left(\frac{t}{8} - r\right)} \left[ \frac{\partial Y_1}{\partial t} + \frac{1}{2t} \left(\lambda - \frac{3}{4}t\right) Y_1 \right] \tag{3.14}$$

then  $Y_1$  and  $Y_2$  satisfy the following system for  $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ :

$$\frac{\partial Y}{\partial t} = AY, \quad A = \begin{pmatrix} -\frac{1}{2t} \left(\lambda - \frac{3}{4}t\right) & \frac{t}{8} - r \\ -\frac{1}{t} & \frac{1}{2t} \left(\lambda - \frac{3}{4}t\right) \end{pmatrix} \tag{3.15}$$

#### 3.1 The Heun Confluent Function

We consider our transformations which are given in equations (3.13) and we relate the generating functions  $F(t, \lambda)$  to the Heun Confluent Function as follows :

**Theorem 3.2.** Equation (3.13) transforms the Riccati equation (2.12) into a linear equation

$$\frac{\partial^2 Y_1}{\partial t^2} = \frac{1}{t-8r} \frac{\partial Y_1}{\partial t} + \left( \frac{\lambda}{2t^2} + \frac{1}{t-8r} \left( \frac{\lambda}{2t} - \frac{3}{8} \right) + \frac{1}{4} \left( \frac{\lambda}{t} - \frac{3}{4} \right)^2 - \frac{1}{t} \left( \frac{t}{8} - r \right) \right) Y_1 \quad (3.16)$$

(a) If  $r = 0$ , the change of dependent variable

$$Y_1 = \left( \frac{1}{4} \right)^{\frac{\lambda}{2} + \frac{3}{2}} t^{\frac{\lambda}{2} + 2} e^{-\frac{t}{8}} G(t, \lambda)$$

simplifies equation (3.16) into the Confluent Hypergeometric equation

$$t \frac{\partial^2 G}{\partial t^2} + \left( \lambda + 3 - \frac{t}{4} \right) \frac{\partial G}{\partial t} + \lambda G = 0 \quad (3.17)$$

where

$$Y_1 = \left( \frac{1}{4} \right)^{\frac{\lambda}{2} + \frac{3}{2}} t^{\frac{\lambda}{2} + 2} e^{-\frac{t}{8}} \left( C_1(\lambda) M\left(-\lambda, \lambda + 3, \frac{t}{4}\right) + C_2(\lambda) U\left(-\lambda, \lambda + 3, \frac{t}{4}\right) \right)$$

such that  $M\left(-\lambda, \lambda + 3, \frac{t}{4}\right)$  and  $U\left(-\lambda, \lambda + 3, \frac{t}{4}\right)$  are Kummer's functions of the first and second kind respectively.

(b) If  $r \neq 0$ , the change of dependent variable

$$Y_1 = t^{1 + \frac{\lambda}{2}} e^{\frac{t}{8}} H(t, \lambda)$$

simplifies equation (3.16) into the Heun Confluent equation

$$\begin{aligned} \frac{\partial^2 H}{\partial t^2} = & \left( \frac{1}{t-8r} - 2 \left( \frac{1}{8} + \frac{1}{t} \left( 1 + \frac{\lambda}{2} \right) \right) \right) \frac{\partial H}{\partial t} + \left( \frac{\lambda}{2t^2} + \frac{1}{t-8r} \left( \frac{\lambda}{2t} - \frac{3}{8} \right) \right. \\ & + \frac{1}{4} \left( \frac{\lambda}{t} - \frac{3}{4} \right)^2 - \frac{1}{t} \left( \frac{t}{8} - r \right) + \frac{1}{t-8r} \left( \frac{1}{8} + \frac{1}{t} \left( 1 + \frac{\lambda}{2} \right) \right) \\ & \left. - \left( \frac{1}{8} + \frac{1}{t} \left( 1 + \frac{\lambda}{2} \right) \right)^2 + \frac{1}{t^2} \left( 1 + \frac{\lambda}{2} \right) \right) Y_1 \end{aligned}$$

where

$$\begin{aligned} Y_1 = & C_1(\lambda) t^{1 + \frac{\lambda}{2}} e^{\frac{t}{8}} Hc\left(2r, 1 + \lambda, -2, r(3 + 3\lambda - 8r), \frac{1}{2}(-1 - 6r)\lambda + 8r^2 + \frac{1}{2}, \frac{t}{8r}\right) \\ & + C_2(\lambda) t^{-\frac{\lambda}{2}} e^{\frac{t}{8}} Hc\left(2r, -1 - \lambda, -2, r(3 + 3\lambda - 8r), \frac{1}{2}(-1 - 6r)\lambda + 8r^2 + \frac{1}{2}, \frac{t}{8r}\right) \end{aligned}$$

## 4 Conclusion

The main results of this paper give another proof of the fact that the  $\sigma_n$ 's are indeed polynomials. Furthermore, following [7], the Hankel determinant and the generation functions which are given above of the fifth Painlevé equations could be used somehow to obtain the Isomonodromic deformations Problem of the fifth Painlevé equations.

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