Stability and Bifurcation Analysis of Discrete Partial Dependent Predator-Prey Model with Delay

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Abstract

In this paper we investigate the dynamics of Euler discretization of predator-prey model with a delay ($\tau$). Its dynamics are studied in term of local stability analysis and the existence of Neimark-Sacker bifurcation. By analyzing the corresponding characteristic equation, it is shown that Neimark-Sacker bifurcation occurs when $\tau$ crosses a sequence of critical values. Such analytical finding is confirmed by some numerical simulations.

Keywords: Discrete predator-prey model, Time delay, Stability, Neimark-Sacker bifurcations.

1 Introduction

The study on the dynamics of predator-prey model is one of dominant subjects in ecology and mathematical ecology due to its universal existence and importance. Time delays have been incorporated into such mathematical models of population dynamics to describe the maturation time, capturing time or other reasons (see [1,5,6,10,12-14]). For a long time, it has been widely investigated
that time delays can have very complicated impact on the dynamics of model, which can cause the loss of stability including the fluctuation of populations. In particular, bifurcation phenomena can occur in parameter-dependent dynamical models. The type of bifurcation that connects the equilibrium with the periodic solution for continuous dynamical models is called Hopf bifurcation.

In Ref. [12], the competition model with a single delay is investigated. By choosing the time delay as the bifurcation parameter, they analyze the stability of interior equilibrium including the existence of Hopf bifurcation for the model. Later, Zhao and Lin [14] modify the model in [12] to describe a partial dependent predator-prey system, i.e.

\[
\begin{align*}
\dot{x}(t) &= x(t)[\eta_1 - a_{11}x(t-\tau) - a_{12}y(t-\tau)] \\
\dot{y}(t) &= y(t)[r_2 + a_{21}x(t-\tau) - a_{22}y(t-\tau)].
\end{align*}
\]

Here \(x(t)\) and \(y(t)\) represent the population densities of prey and predator at time \(t\), respectively; while \(\eta_1, a_{11}, a_{12}, r_2, a_{21}, a_{22}, \tau\) are all positive constants. Zhao and Lin [14] observe that a bifurcating phenomenon can still occur in the resulted predator-prey model where the bifurcation is also controlled by a time delay.

Considering the need of scientific computation and real time simulation, we are interested in the behavior of a discrete model. Discrete models generally have more dynamical complexity than the corresponding continuous models. In other words, discrete models may show richer dynamics than the continuous ones. Many researchers have considered numerical approximation of a scalar delay differential equation by using different numerical methods, such as nonstandard finite-different method, Euler method, Runge-Kutta method (see [1, 3-9, 11, 13]). Specifically, in this paper, we implement the Euler method to discretize the following delay differential equations (1) and study the dynamical behaviors of the obtained discrete model. We mainly focus on the existence of Neimark-Sacker bifurcation, which is the discrete analogue of Hopf bifurcation.

2 Discrete Model: equilibria and their stability

Before discretize model (1), we first denote \(u(t) = x(t\tau), v(t) = y(t\tau)\) such that equation (1) can be rewritten as

\[
\begin{align*}
\frac{du(t)}{dt} &= \eta u(t)[\eta_1 - a_{11}u(t-1) - a_{12}v(t-1)], \\
\frac{dv(t)}{dt} &= \eta v(t)[r_2 + a_{21}u(t-1) - a_{22}v(t-1)].
\end{align*}
\]

Then we apply the Euler method to get a delay difference equation
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\[ u_{n+1} = u_n + \tilde{\theta} u_n [\eta - a_{11} u_{n-m} - a_{12} v_{n-m}], \]
\[ v_{n+1} = v_n + \tilde{\theta} v_n [r_2 + a_{21} u_{n-m} - a_{22} v_{n-m}], \]

(3)

where \( u_n \) is an approximate value to \( u(nh) \) and \( \tilde{\theta} = 1/m \) where \( m \in \mathbb{Z}^+ \) is the step size. It can easily be shown that system (3) has the following equilibria

(i) \( E_0 = (0,0) \) is trivial equilibrium,
(ii) \( E_1 = (0,r_2/a_{22}) \) is axial equilibrium in the absence of prey \( (u = 0) \),
(iii) \( E_2 = (\tilde{\eta}/a_{11},0) \) is axial equilibrium in the absence of predator \( (v = 0) \),
(iv) \( E^* = (u^*,v^*) \) is interior equilibrium, where \( u^* = (a_{22}\tilde{\eta} - a_{12} r_2)/\Delta \), \( v^* = (a_{21}\tilde{\eta} + a_{11} r_2)/\Delta \) and \( \Delta = a_{11} a_{22} + a_{12} a_{21} \).

It is observed that the first of three equilibria always exist while the interior equilibrium \( E^* = (u^*,v^*) \) exists only if \( a_{22}\tilde{\eta} - a_{12} r_2 > 0 \).

To study the stability of an equilibrium \((\tilde{u},\tilde{v})\), we first linearize system (3) by substituting \( \tilde{u}_n = u_n - \tilde{u} \) and \( \tilde{v}_n = v_n - \tilde{v} \), with \( \tilde{u}_n << 1 \) and \( \tilde{v}_n << 1 \), to get

\[ \tilde{u}_{n+1} = [1 + (\eta - a_{11}\tilde{u} - a_{12}\tilde{v})\tilde{\theta}][\tilde{u}_n - a_{11}\tilde{u}\tilde{\theta}\tilde{u}_{n-m} - a_{12}\tilde{u}\tilde{\theta}\tilde{v}_{n-m}], \]
\[ \tilde{v}_{n+1} = a_{21}\tilde{v}\tilde{\theta}\tilde{u}_{n-m} + [1 + (r_2 + a_{21}\tilde{u} - a_{22}\tilde{v})\tilde{\theta}][\tilde{v}_n - a_{22}\tilde{v}\tilde{\theta}\tilde{v}_{n-m}]. \]

(4)

System of equations (4) can be written as

\[ Z_{n+1} = AZ_n, \]

(5)

where

\[
A = \begin{pmatrix}
 a & 0 & \cdots & 0 & b & 0 & \cdots & 0 & c \\
 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & d & e & \cdots & 0 & f \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix},
\]

(6)

with \( Z_n = (\tilde{u}_n,\tilde{u}_{n-1},\ldots,\tilde{u}_{n-m},\tilde{v}_n,\tilde{v}_{n-1},\ldots,\tilde{v}_{n-m})^T \), \( a = 1 + (\eta - a_{11}\tilde{u} - a_{12}\tilde{v})\tilde{\theta} \), \( b = -a_{11}\tilde{u}\tilde{\theta} \), \( c = -a_{12}\tilde{u}\tilde{\theta} \), \( d = a_{21}\tilde{v}\tilde{\theta} \), \( e = 1 + (r_2 + a_{21}\tilde{u} - a_{22}\tilde{v})\tilde{\theta} \), and \( f = -a_{22}\tilde{v}\tilde{\theta} \). The characteristic equation of \( A \) is given by
\[ \lambda^{2m+2} + A_1 \lambda^{2m+1} + A_2 \lambda^{2m} + A_3 \lambda^{m+1} + A_4 \lambda^m + A_5 = 0, \quad (7) \]

where \( A_1 = -2, \ A_2 = 1, \ A_3 = (a_{11} \dot{u} + a_{22} \dot{v}) \theta h, \) and \( A_4 = -(a_{11} \dot{u} + a_{22} \dot{v}) \theta h. \) It is well known that the stability of equilibrium \((\dot{u}, \dot{v})\) depends on the distribution of the zeros of the roots of (7). In the following we study the stability of each equilibrium of map (3).

### 2.1 Trivial equilibrium \( E_0 \)

By substituting equilibrium \( E_0 = (0,0) \) into equation (7), we have the following characteristic equation

\[ \lambda^{2m} (\lambda - (1 + n_1 \theta h))(\lambda - (1 + r_2 \theta h)) = 0. \]

This equation has roots \( \lambda_1 = (1 + r_1 \theta h), \ \lambda_2 = (1 + r_2 \theta h) \) and \( 2m \)-fold roots \( \lambda = 0. \) It is clearly that \( |\lambda_1| > 1 \) and \( |\lambda_2| > 1, \) and hence we conclude that the equilibrium \( E_0 = (0,0) \) is unstable.

### 2.2 Axial equilibrium in the absence of prey \( E_1 \)

The characteristic equation (7) at equilibrium \( E_1 = (0, r_2 / a_{22}) \) is

\[ \lambda^m \left( \lambda - \left(1 + \left( n - \frac{a_{12} r_2}{a_{22}} \right) \theta h \right) \right) \left( \lambda^{m+1} - \lambda^m + r_2 \theta h \right) = 0. \quad (8) \]

To analyze the roots of this characteristic equation, we apply the following lemma.

**Lemma 1.** (Elaydi [2]). The zero solution of

\[ z(n + m - 1) - z(n + m) + q z(n) = 0 \quad (9) \]

is asymptotically stable if and only if \( 0 < q < 2 \cos \left( \frac{2 \pi}{2m + 1} \right). \)

It is clear that equation (8) has root \( \lambda_1 = \left(1 + \left( n - \frac{a_{12} r_2}{a_{22}} \right) \theta h \right), \) \( m \)-fold roots \( \lambda = 0 \) and other roots which are determined by equation \( \lambda^{m+1} - \lambda^m + r_2 \theta h = 0. \) \( |\lambda_1| < 1 \) if and only if \( a_{22} r_1 - a_{12} r_2 < 0 \) and \( h < h_1 = \frac{2 a_{22}}{(a_{12} r_2 - a_{22} r_1) r}. \) Equation \( \lambda^{m+1} - \lambda^m + r_2 \theta h = 0 \) corresponds to difference equation (9) and therefore, using
Lemma 1, all modulus of its roots are less than one if and only if
\[ h < h_2 = 2\cos\left(\frac{m\pi}{2m+1}\right)/r_2\tau. \]
Hence, equilibrium \( E_1 \) is asymptotically stable if and only if
\[ a_{22}r_1 - a_{12}r_2 < 0 \quad \text{and} \quad h < \min\{h_1, h_2\}. \]

### 2.3 Axial equilibrium in the absence of predator \( E_2 \)

At equilibrium \( E_2 = (r_1/a_1, 0) \), the characteristic equation (7) can be written as
\[
\lambda^m \left[ \lambda - \left( 1 + \left( r_2 + \frac{a_{21}r_1}{a_{11}} \right) \tau \right) \right] \left[ \lambda^{m+1} - \lambda^m + r_1 \tau \right] = 0.
\]
One of its roots is \( \lambda_1 = \left( 1 + \left( r_2 + \frac{a_{21}r_1}{a_{11}} \right) \tau \right) > 1 \) and therefore equilibrium \( E_2 = (r_1/a_1, 0) \) is unstable.

### 2.4 Interior equilibrium \( E^* \)

The stability of equilibrium \( E^* = (u^*, v^*) \) is analyzed by employing the results of Zhang et al. [11].

**Lemma 2.** Zhang et al. [11]. Suppose that \( B \in \mathcal{R} \) is a bounded closed and connected set, \( f(\lambda, \tau) = \lambda^m + p_1(\tau)\lambda^{m-1} + \cdots + p_m(\tau) \) is continuous in \( (\lambda, \tau) \in C \times B \), and \( \tau \) is a parameter, \( \tau \in B \). Then as \( \tau \) varies, the sum of the order of the zeros of \( f(\lambda, \tau) \) out of the unit circle \( \{ \lambda \in C : |\lambda| > 1 \} \) can change only if a zero appears on or crosses the unit circle.

**Lemma 3.** There exists \( \tau_0 > 0 \) such that for \( 0 < \tau < \tau_0 \) all roots of (7) have modulus less than one.

**Proof.** When \( \tau = 0 \), the characteristic equation (7) at \( E^* \) is
\[
\lambda^{2m+2} - 2\lambda^{2m+1} + \lambda^{2m} = 0.
\]
Equation (10) has \( 2m \)-fold roots \( \lambda = 0 \) and a double root \( \lambda = 1 \), at \( \tau = 0 \).

Consider the root \( \lambda(\tau) \) such that \( |\lambda(0)| = 1 \). This root depends continuously on \( \tau \) and is a differentiable function of \( \tau \). From equation (7), we find that \( \frac{d\lambda}{d\tau} \) satisfies the following equation.
Thus, if
\[
(a_1 u^* + a_{22} v^*)^2 h^2 - 4(a_1 a_{22} + a_{12} a_{21}) u^* v^* h^2 = 0.
\]
then we have
\[
\frac{d\lambda}{d\tau} \bigg|_{r=0, \delta=1} = -\frac{(a_1 u^* + a_{22} v^*)h}{2},
\]
and
\[
\frac{d|\lambda|^2}{d\tau} \bigg|_{r=0, \delta=1} = -(a_1 u^* + a_{22} v^*)h < 0.
\]
Consequently, $|\lambda| < 1$ for all sufficiently small $\tau > 0$. Thus all roots of equation (7) lie in $|\lambda| < 1$ for sufficiently small positive $\tau > 0$, and the existence of the maximal $\tau$ follows. The proof is completed.$\square$

A Neimark-Sacker bifurcation occurs when a complex conjugate pair of eigenvalues of $A$ cross the unit circle as $\tau$ varies. We will determine the value of $\tau$ such that there exist roots on the unit circle. Let $\lambda = e^{i\omega}$ be a root of (7) when $\tau = \tau^*$. Then
\[
e^{i(2m+2)\omega} + A_1^* e^{i(2m+1)\omega} + A_2^* e^{i(2m)\omega} + A_3^* e^{i(m+1)\omega} + A_4^* e^{im\omega} + A_5^* = 0, \tag{11}
\]
where $A_j^*, j=1,2,...,5$ is the value of $A_j$ at $\tau = \tau^*$. Separating the real and imaginary parts, we have
\[
\cos(2m+2)\omega^* + A_1^* \cos(2m+1)\omega^* + A_2^* \cos 2m\omega^* + A_3^* \cos(m+1)\omega^* + A_4^* \cos m\omega^* = -A_5^* \\
\sin(2m+2)\omega^* + A_1^* \sin(2m+1)\omega^* + A_2^* \sin 2m\omega^* + A_3^* \sin(m+1)\omega^* + A_4^* \sin m\omega^* = 0. \tag{12}
\]
From equation (12), there exists an infinite sequence of value of the time delay parameter $0 < \tau_0 < \tau_1 < ... < \tau_i < ...$ satisfying (12).

Lemma 4. If the step size $h$ is sufficiently small, then
\[ d_h = \frac{d|\lambda|^2}{d\tau} \bigg|_{\tau^{*}, \omega^{*}} > 0. \]

**Proof.** We can rewrite (7) at \( E^{*} \) in the form

\[ \lambda^{m+1} + \lambda^{m} + (a_{11}u^{*} + a_{22}v^{*})h = -\left(\frac{a_{11}a_{22} + a_{12}a_{21}}{\lambda^{m} (\lambda - 1)}\right) u^{*} v^{*} \tau^{2} h^{2} = o(h^{2}). \]  

(13)

For sufficiently small \( h \), the right hand side of equation (13) can be neglected and hence we have the following equation

\[ \lambda^{m+1} + \lambda^{m} + (a_{11}u^{*} + a_{22}v^{*})h = 0. \]  

(14)

Substituting \( \lambda = e^{i\omega} \) into equation (14) leads to

\[ e^{i(m+1)\omega^{*}} + e^{im\omega^{*}} + (a_{11}u^{*} + a_{22}v^{*})h = 0. \]  

(15)

The real and imaginary parts of equation (15) are respectively

\[ \cos((m+1)\omega^{*}) - \cos(m\omega^{*}) = -(a_{11}u^{*} + a_{22}v^{*})h, \]

\[ \sin((m+1)\omega^{*}) - \sin(m\omega^{*}) = 0. \]  

(16)

Based on equations (14) and (16), we can show that

\[ d_h = \frac{d|\lambda|^2}{d\tau} \bigg|_{\tau^{*}, \omega^{*}} = -\left(\frac{(a_{11}u^{*} + a_{22}v^{*})h}{(m+1)\cos(m+1)\omega^{*} - 2m \cos m \omega^{*}} \right) \cdot \frac{2(m+1)\cos(m+1)\omega^{*} - 2m \cos m \omega^{*}}{(m+1)^2 + m^2 - 2m(m+1)\cos \omega^{*}}. \]

From equation (16), it is found that

\[ \cos \omega^{*} = 1 - \frac{(\frac{(a_{11}u^{*} + a_{22}v^{*})h}{2})^2}{2}; \cos m \omega^{*} = \frac{(a_{11}u^{*} + a_{22}v^{*})h}{2}; \]

\[ \cos((m+1)\omega^{*}) = -\frac{(a_{11}u^{*} + a_{22}v^{*})h}{2}, \]

and it is easy to see that \((m+1)^2 + m^2 - 2m(m+1)\cos \omega^{*} > 0\). Hence \( d_h > 0 \) and the proof is completed. \( \Box \)

Based on the previous discussion, we can conclude the following theorem.
Theorem 5. Assume that the condition (12) holds. There exists a sequence of values of time delay parameter $0 < \tau_0 < \tau_1 < ... < \tau_i < ...$, such that we have: (i) the interior equilibrium of system (2.2) is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$ (ii) the interior equilibrium of system (3) undergoes a Neimark-Sacker bifurcation when $\tau = \tau_i$, $i = 0,1,2,...$, where $\tau_i$ satisfies equation (12).

3 Numerical Simulations

In this section, we will confirm and illustrate the analytical results by numerical simulations. Let us consider the following particular case of system (3).

Case 1: By choosing that $\eta = 0.4, r_2 = 1, a_{11} = 1, a_{12} = 1, a_{21} = 2, a_{22} = 1, m = 20$, then system (3) becomes

$$
\begin{align*}
    u_{n+1} &= u_n + \tau h u_n [0.4 - u_{n-m} - v_{n-m}] \\
    v_{n+1} &= v_n + \tau h v_n [1 + 2u_{n-m} - v_{n-m}].
\end{align*}
$$

System (17) has equilibrium $E_0 = (0,0), E_1 = (0,1), \text{ and } E_2 = (0.4,0)$. In Figure 1, we show the numerical solution (phase plot and waveform plot) of (17) with initial value $u_0 = 0.4$, $v_0 = 0.8$; $n = 1,2,...,21$, $h = 0.05$ and $\tau = 1.5$. Since $a_{22}r_1 - a_{12}r_2 = -0.6 < 0$ and $h = 0.05 < \min\{h_1 = 3.334, h_2 = 0.051\}$, equilibrium $E_1 = (0,1)$ of (17) is asymptotically stable as seen in Figure 1.

![Figure 1](image)

Figure 1. Numerical solution of system (17) with initial value $u_0 = 0.4$, $v_0 = 0.8$; $n = 1,2,...,21$, $h = 0.05$ and $\tau = 1.5$: (a) phase plot; (b) waveform plot.

Case 2: If we take $\eta = 1, r_2 = 0.4, a_{11} = 1, a_{12} = 1, a_{21} = 2, a_{22} = 1, m = 20$, then system (3) can be written as
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\[ u_{n+1} = u_n + \tau h u_n [1 - u_{n-m} - v_{n-m}] \]
\[ v_{n+1} = v_n + \tau h v_n [0.4 + 2u_{n-m} - v_{n-m}] \]

System (18) has an interior equilibrium \( E^* = (u^*, v^*) = (0.2, 0.8) \). Using equation (12), we find that the critical delay for Neimark-Sacker bifurcation \( \tau_0 \approx 1.13525 \).

In Figure 2, we show the numerical solution of (18) using \( h = 0.05 \) for different values of \( \tau \). Figure 2a shows that using initial value \( u_n = 0.1, v_n = 0.2; n = 1, 2, ..., 21 \), the solution is convergent to equilibrium \( E^* = (0.2, 0.8) \) if we take \( \tau = 1.1 < \tau_0 \approx 1.13525 \). If we use bigger time delay which is larger than the critical value, i.e. \( \tau = 1.145 > \tau_0 \approx 1.13525 \) then the solution oscillates periodically; see Figure 2b. This shows that a Neimark-Sacker bifurcation occurs when \( \tau \) varies and passes through \( \tau_0 \approx 1.13525 \). Furthermore, using the same time delay \( \tau = 1.145 > \tau_0 \approx 1.13525 \) but with initial value which is closer to \( E^* (0.2, 0.8) \), i.e. \( u_n = 0.21, v_n = 0.81, n = 1, 2, ..., 21 \), then the solution is also attracted to the limit cycle as in Figure 2b. Hence the Neimark-Sacker bifurcation is supercritical.

Figure 2. Numerical solution of system (18) with \( h = 0.05 \) and (a) \( \tau = 1.1 \) and 
\( u_n = 0.1, v_n = 0.2; n = 1, 2, ..., 21 \); (b) \( \tau = 1.145 \) and 
\( u_n = 0.1, v_n = 0.2; n = 1, 2, ..., 21 \); (c) \( \tau = 1.145 \) and 
\( u_n = 0.21, v_n = 0.81, n = 1, 2, ..., 21 \).

4 Conclusion

The dynamics of numerical discretization of predator-prey model with delay have been investigated. In particular we have established the stability properties of
the existing equilibria including the existence of Neimark-Sacker bifurcation controlled by time delay. Such analytical results have been confirmed numerically.

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