A Buffon Type Problem for Two Delone Trapezoidal Lattices with Obstacles (Four Trapetium)

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Abstract

In this paper we consider a Delone lattices with the cell represented in the figure 1 and we compute the probability that a segment of random position and constant length intersects a side of the lattice.

Keywords: Geometric Probability, stochastic geometry, random sets, random convex sets and integral geometry

1 Main Results

Let $\mathcal{R}_1 (a, b, \alpha; m)$ be the Delone lattice with fundamental cell $C_0^{(1)}$ represented in the figure
where \( a \leq b \) and \( \alpha \in \left[ \frac{\pi}{4}, \frac{\pi}{2} \right] \) an angle. The eight obstacles are isosceles triangles of sides \( \frac{m}{2}, \frac{m}{2}, \) with \( 0 \leq m < \frac{b-a}{2 \cos \alpha} \).

From the figure 1 we have the follow relations

\[
|AH| = |HD| = a, \quad |BG| = |GC| = b,
\]

\[
|AE| = |EB| = \frac{b-a}{2 \cos \alpha}, \quad |GL| = |HL| = \frac{(b-a)\tan \alpha}{2},
\]

\[
|EL| = |FL| = \frac{a+b}{2}, \quad (1)
\]

\[
|A_1A_2| = |E_1E_3| = |D_1D_2| = |F_1F_3| = m \cos \frac{\alpha}{2},
\]

\[
|B_1B_2| = |E_1E_2| = |C_1C_2| = |F_1F_2| = m \sin \frac{\alpha}{2}, \quad (2)
\]

\[
\text{area}AA_1A_2 = \text{area}EE_1E_3 = \text{area}DD_1D_2 = \text{area}FF_1E_3 =
\]

\[
\text{area}BB_1B_2 = \text{area}CC_1C_2 = \text{area}EE_1E_2 = \text{area}FF_1E_2 = \frac{m^2}{8} \sin \alpha, \quad (3)
\]

\[
\text{area}C_{01}^{(1)} = \text{area}C_{02}^{(1)} = \frac{(3a+b) \tan \alpha}{8} - \frac{m^2}{4} \sin \alpha,
\]

\[
\text{area}C_{03}^{(1)} = \text{area}C_{04}^{(1)} = \frac{(a+3b) \tan \alpha}{8} - \frac{m^2}{4} \sin \alpha. \quad (4)
\]

Now we consider a segment \( s \) of random position and of constant length \( l < \min \left( \frac{b-a}{2 \cos \alpha} - m, \frac{a+b}{2} - \frac{m}{2} \right) \), we want to compute the probability that
this segment intersects a side of lattice $R_1$, therefore the probability $P_{\text{int}}^{(1)}$ that the segment $s$ intersects the side of the fundamental cell $C_0^{(1)}$.

The position of the segment $s$ is determined by his middle point $O$ and by the angle $\varphi$ that it forms with the side $BC$ of the cell $C_0^{(1)}$.

To compute the probability $P_{\text{int}}^{(1)}$ we consider the limit positions of the segment $s$, for an assigned value of $\varphi$, situated in the cell $C_0^{(1)}$, ($i=1, 2, 3, 4$).

We denote with $\widehat{C}_0^{(1)}(\varphi)$ the polygon determined from these positions we have the figure

![Diagram](image)

From here we have

$$area\widehat{C}_0^{(1)}(\varphi) = area\widehat{C}_0^{(2)}(\varphi) =$$

$$areaC_0^{(1)} - [areaa_1(\varphi) + areaa_2(\varphi) + \ldots + areaa_7(\varphi)], \quad (5)$$

$$area\widehat{C}_0^{(3)}(\varphi) = area\widehat{C}_0^{(4)}(\varphi) = areaC_0^{(1)} - [areab_1(\varphi) + areab_2(\varphi) + \ldots + areab_7(\varphi)]. \quad (6)$$

The figure
$$
\begin{align*}
\overrightarrow{AA_1A_2} &= \overrightarrow{AA_2A_1} = \frac{\alpha}{2}, \quad \overrightarrow{AA_2A_3} = \varphi, \\
\overrightarrow{A_2A_1A_3} &= \pi - \frac{\alpha}{2}, \quad \overrightarrow{A_1A_2A_3} = \varphi - \frac{\alpha}{2}, \quad \overrightarrow{A_1A_3A_2} = \alpha - \varphi. \quad (7)
\end{align*}
$$

From the triangle $A_1A_2A_3$ follow that:

$$
\frac{|A_1A_2|}{\sin (\alpha - \varphi)} = \frac{l}{\sin \frac{\alpha}{2}} = \frac{|A_1A_3|}{\sin \left(\varphi - \frac{\alpha}{2}\right)},
$$

therefore, with the (2),

$$
\frac{m \cos \frac{\alpha}{2}}{\sin (\alpha - \varphi)} = \frac{l}{\sin \frac{\alpha}{2}} = \frac{|A_1A_3|}{\sin \left(\varphi - \frac{\alpha}{2}\right)}.
$$

From here we have

$$
|A_1A_3| = \frac{l \sin \left(\varphi - \frac{\alpha}{2}\right)}{\sin \frac{\alpha}{2}} \quad (8)
$$

and the condition

$$
2l \sin (\alpha - \varphi) = m \sin \alpha, \quad (m \neq 0). \quad (9)
$$

Moreover we have

$$
h_1 = |A_1A_2| \sin \overrightarrow{A_1A_2A_3} = m \cos \frac{\alpha}{2} \sin \left(\varphi - \frac{\alpha}{2}\right),
$$
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therefore

\[
areaa_1 (\varphi) = \frac{l h_1}{2} = \frac{m l}{2} \cos \frac{\alpha}{2} \sin \left( \varphi - \frac{\alpha}{2} \right).
\]

Considering the relation (9), follow

\[
areaa_1 (\varphi) = \frac{l^2}{2 \sin \frac{\alpha}{2}} \sin \left( \varphi - \frac{\alpha}{2} \right) \sin (\alpha - \varphi).
\] (10)

Now we consider the figure

\[
\begin{align*}
\text{fig. 4}
\end{align*}
\]

With the (7) we have

\[
h_2 = \frac{l}{2} \sin A_3 E_2 E_3 = \frac{l}{2} \sin (\alpha - \varphi)
\]

and with the (8)

\[
|A_3 E_2| = \frac{b - a}{2 \cos \alpha} - m - \frac{l \sin \left( \varphi - \frac{\alpha}{2} \right)}{\sin \frac{\alpha}{2}}.
\]

Hence

\[
areaa_2 (\varphi) = \left[ \frac{b - a}{2 \cos \alpha} - m - \frac{l \sin \left( \varphi - \frac{\alpha}{2} \right)}{\sin \frac{\alpha}{2}} \right] \cdot \frac{l}{2} \sin (\alpha - \varphi).
\] (11)

The relations (10) and (11) give us

\[
areaa_1 (\varphi) + areaa_2 (\varphi) = \left( \frac{b - a}{2 \cos \alpha} - m \right) \cdot \frac{l}{2} \sin (\alpha - \varphi).
\] (12)

To compute \( areaa_3 (\varphi) \), we consider the figure
We have
\[ E_2 E_1 E_4 = \pi - \left( \frac{\pi}{2} - \frac{\alpha}{2} + \phi \right) = \frac{\pi}{2} - \phi + \frac{\alpha}{2}, \]
\[ h_3 = \frac{l}{2} \sin \left( \frac{\pi}{2} - \phi + \frac{\alpha}{2} \right) = \frac{l}{2} \cos \left( \phi - \frac{\alpha}{2} \right), \]
and consequently, with the (2),
\[ \text{area}_{3}(\phi) = \frac{ml}{2} \sin \frac{\alpha}{2} \cos \left( \phi - \frac{\alpha}{2} \right). \] (13)

The figure
\[ \text{fig.5} \]

\[ |LL_1| = l \sin \phi, \quad |LL_2| = l \cos \phi, \quad \hat{L_2L_1L} = \frac{\pi}{2} - \varphi, \] (14)
therefore
\[ \text{area}_{5}(\phi) = \frac{l^2 \sin \varphi \cos \varphi}{2}. \] (15)

The figure
\[ \text{fig.6} \]
and the second relation (14) give us

\[ h_4 = \frac{l}{2} \sin \varphi, \quad |E_1L_2| = \frac{a + b}{2} - \frac{m}{2} - l \cos \varphi. \]

Therefore

\[ \text{area}a_4(\varphi) = \left( \frac{a + b}{2} - \frac{m}{2} - l \cos \varphi \right) \cdot \frac{l}{2} \sin \varphi. \]  

(16)

From the relations (15) and (16) follow

\[ \text{area}a_4(\varphi) + \text{area}a_5(\varphi) = (a + b - m) \cdot \frac{l}{4} \sin \varphi. \]  

(17)

Now we consider the figure

\[ \hat{H}'L'L_1 = \hat{L_2L_1L} = \frac{\pi}{2} - \varphi, \]
\[ h_6 = \frac{l}{2} \sin \left( \frac{\pi}{2} - \varphi \right) = \frac{l}{2} \cos \varphi, \]

\[ |HL_1| = |HL| - |LL_1| = \frac{(b - a) \tan \alpha}{2} - l \sin \varphi, \]

therefore

\[ \text{area}_{a_6}(\varphi) = \left[ \frac{(b - a) \tan \alpha}{2} - l \sin \varphi \right] \cdot \frac{l}{2} \cos \varphi. \quad (18) \]

At the end considering the figure

\[ \text{fig.9} \]

follow

\[ h_7 = \frac{l}{2} \sin \varphi, \quad |A_2H| = a - \frac{m}{2} \]

hence

\[ \text{area}_{a_7}(\varphi) = (2a - m) \cdot \frac{l}{4} \sin \varphi. \quad (19) \]

Replacing in the relation (5) the expressions (12), (13), (17), (18) and (19) we obtain

\[ \text{area}^{(1)}_{C_{01}}(\varphi) = \text{area}^{(1)}_{C_{02}}(\varphi) = \text{area}^{(1)}_{C_{01}} - \left\{ [(b - a) \tan \alpha - | \right. \]

\[ \left. \frac{m}{2} \sin \varphi \right\} \cdot \frac{l}{2} \cos \varphi + \left[ 2a + \frac{m}{2} (\cos \alpha - 2) \right] \cdot \frac{l}{2} \sin \varphi - \frac{l^2}{4} \sin 2\alpha \}. \quad (20) \]

Now we compute \( \text{area}_{b_i}(\varphi), \ (i = 1, 2, \ldots, 7) \).

From the figure 2 follow

\[ \text{area}_{b_1}(\varphi) = \text{area}_{a_1}(\varphi), \quad \text{area}_{b_2}(\varphi) = \text{area}_{a_2}(\varphi), \]

\[ \text{area}_{b_3}(\varphi) = \text{area}_{a_3}(\varphi), \quad \text{area}_{b_5}(\varphi) = \text{area}_{a_5}(\varphi), \]

\[ \text{area}_{b_6}(\varphi) = \text{area}_{a_6}(\varphi). \quad (21) \]
We have

$$|B_1G_2| = |BG| - \frac{m}{2} - |GG_2| = b - \frac{m}{2} - l \cos \varphi =$$

$$\frac{a+b}{2} + \frac{b-a}{2} - \frac{m}{2} - l \cos \varphi,$$

therefore, considering the relation (16),

$$area_{b_4}(\varphi) = \left(\frac{a+b}{2} - \frac{m}{2} - l \cos \varphi + \frac{b-a}{2}\right) \cdot \frac{l}{2} \sin \varphi =$$

$$area_{a_4}(\varphi) + \frac{(b-a)l}{4} \cdot \sin \varphi$$

(22)

At the same way we have

$$|E_1L| = |EL| - \frac{m}{2} = \frac{a+b}{2} - \frac{m}{2} = a - \frac{m}{2} + \frac{b-a}{2},$$

hence, with the (19),

$$area_{b_7}(\varphi) = \left(\frac{a}{2} - \frac{m}{2} + \frac{b-a}{2}\right) \cdot \frac{l}{2} \sin \varphi =$$

$$area_{a_7}(\varphi) + \frac{(b-a)l}{4} \cdot \sin \varphi.$$  

(23)

The relations (21), (22) and (23) give us

$$area_{b_1}(\varphi) + area_{b_2}(\varphi) + \ldots area_{b_7}(\varphi) =$$

$$area_{a_1}(\varphi) + area_{a_2}(\varphi) + \ldots area_{a_7}(\varphi) + \frac{(b-a)l}{2} \cdot \sin \varphi.$$  

(24)

From the relations (5), (6) and (24) follow

$$area\tilde{C}_{03}^{(i)}(\varphi) = area\tilde{C}_{04}^{(i)}(\varphi) = areaC_{03}^{(i)} -$$

$$\left[area_{a_1}(\varphi) + area_{a_2}(\varphi) + \ldots area_{a_7}(\varphi) + \frac{(b-a)l}{2} \cdot \sin \varphi\right] =$$

$$areaC_{03}^{(i)} - \left[areaC_{01}^{(i)} - area\tilde{C}_{01}^{(i)}(\varphi) + \frac{(b-a)l}{2} \cdot \sin \varphi\right] =$$

$$area\tilde{C}_{01}^{(i)}(\varphi) + areaC_{03}^{(i)} - areaC_{01}^{(i)} - \frac{(b-a)l}{2} \cdot \sin \varphi.$$  

(25)

Denoting with $M_i^{(1)}, (i = 1, 2, 3, 4)$, the set of segments $s$ which have the middle point in $C_{0i}^{(i)}$ and with $N_i^{(1)}$ the set of segments $s$ completely contained in $C_{0i}^{(i)}$, we have [2]:
$P^{(1)}_{\text{int}} = 1 - \frac{\mu \left( N_1^{(1)} \right) + \mu \left( N_2^{(1)} \right) + \mu \left( N_3^{(1)} \right) + \mu \left( N_4^{(1)} \right)}{\mu \left( M_1^{(1)} \right) + \mu \left( M_2^{(1)} \right) + \mu \left( M_3^{(1)} \right) + \mu \left( M_4^{(1)} \right)}, \quad (26)$

where $\mu$ is the Lebesgue measure in Euclidean plane.

To compute the measures we use the Poincaré kinematic measure [1]:

$$dK = dx \wedge dy \wedge d\varphi,$$

where $x, y$ are the coordinates of the point $O$ and $\varphi$ the defined angle.

From the figure

![Figure 10](image_url)

follow

$$\varphi_1 = 0, \quad \varphi_2 = \alpha, \quad \varphi_1 \leq \varphi \leq \varphi_2,$$

therefore

$$\varphi \in [0, \alpha]. \quad (27)$$

Considering the formulas (20) and (27) we can write

$$\mu \left( M_1^{(1)} \right) = \mu \left( M_2^{(1)} \right) = \int_{0}^{\alpha} d\varphi \iint_{\{ (x, y) \in C_{01}^{(1)} \}} dxdy =$$

$$\int_{0}^{\alpha} \left[ \text{area}C_{01}^{(1)} \right] d\varphi = aareaC_{01}^{(1)}, \quad (28)$$

$$\mu \left( N_1^{(1)} \right) = \mu \left( N_2^{(1)} \right) = \int_{0}^{\alpha} d\varphi \iint_{\{ (x, y) \in \tilde{C}_{01}^{(1)}(\varphi) \}} dxdy = \int_{0}^{\alpha} \left[ \text{area}\tilde{C}_{01}^{(1)}(\varphi) \right] d\varphi =$$

$$aareaC_{01}^{(1)} - \frac{l}{2} \left[ \left( 1 - \cos \alpha \right) \left( 1 - \frac{1}{\cos \alpha} \right) +$$
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\[ b \left(1 + \frac{1}{\cos \alpha}\right) - \frac{m}{2} (3 + \cos \alpha) \} + (1 + \cos 2\alpha) \frac{l^2}{8}. \] (29)

At the same way we have

\[ \mu \left(M^{(1)}_3\right) = \mu \left(M^{(1)}_4\right) = \int_0^\alpha d\phi \int \int_{\{(x,y)\in C^{(1)}_{03}\}} dxdy = \]

\[ \int_0^\alpha \left[ areaC^{(1)}_{03} \right] d\phi = \alpha areaC^{(1)}_{03}. \] (30)

The formula (25) give us

\[ \mu \left(N^{(1)}_3\right) = \mu \left(N^{(1)}_4\right) = \int_0^\alpha d\phi \int \int_{\{(x,y)\in C^{(1)}_{03}(\phi)\}} dxdy = \]

\[ \int_0^\alpha \left[ areaC^{(1)}_{03}(\phi) \right] d\phi = \int_0^\alpha \left[ areaC^{(1)}_{01}(\phi) + areaC^{(1)}_{03} - areaC^{(1)}_{01} \right] \]

\[ = \alpha \left[ areaC^{(1)}_{03} - areaC^{(1)}_{01} \right] - \frac{(b-a)l}{2} (1 - \cos \alpha). \] (31)

Replacing the expressions (28), (29), (30), and (31) in the (26) we obtain

\[ P^{(1)}_{\text{int}} = \frac{1}{2\alpha \left[ areaC^{(1)}_{01} + areaC^{(1)}_{03} \right]} \left( \frac{l}{2} \left\{ (1 - \cos \alpha) \cdot \right. \right. \]

\[ \left. \left[ a \left(1 - \frac{1}{\cos \alpha}\right) + b \left(1 + \frac{1}{\cos \alpha}\right) \right] - \frac{m}{2} (3 + \cos \alpha) \left\} - \right. \]

\[ \sin^2 \alpha \cdot l^2 - 2\alpha areaC^{(1)}_{01} - (b-a) (1 - \cos \alpha) \right). \]

For \( m = 0 \), therefore when the obstacles becomes points, this probability becomes

\[ P^{(1)} = \frac{1}{(b^2 - a^2)\alpha \tan \alpha} \left\{ (1 - \cos \alpha) \left[ a \left(1 - \frac{1}{\cos \alpha}\right) + \right. \right. \]

\[ b \left(1 + \frac{1}{\cos \alpha}\right) \right] \cdot \frac{l}{2} - \sin^2 \alpha \cdot l^2 - \]
(b - a) (1 - \cos \alpha) - \frac{(3a + b)(b - a)}{4} \cdot \alpha \tan \alpha \right) .

If, for example \alpha = \frac{\pi}{3} this probability is written

\[ P = \frac{\sqrt{3}}{\pi (b^2 - a^2)} \left\{ \frac{3b - a}{4} \cdot l - \frac{3}{4} l^2 + \frac{b - a}{4} \left[ 4 - (3a + b) \sqrt{3}\pi \right] \right\} . \]

References


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