Approximate Calculation of the Multiple Integrals’ Value by Repeated Application of Gauss and Simpson’s Quadrature Formulas

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Abstract. The work deals with the construction of multidimensional quadrature formulas for the approximate calculation of the multiple integrals’ values by repeated application of Gauss and Simpson’s quadrature formulas. We’ve proved the quadrature formulas’ correctness.

Keywords: multidimensional cubature formula, there-use method, multiple integrals

1. Introduction

This paper is devoted to forming the repeated cubature formulas for n-fold integrals’ values calculation by means of repeated application of quadrature Simpson's and Gauss' formulas.

The method of quadrature formulas’ re-use.

As we know from the analysis, multiple integrals can be computed by the single integrals’ re-calculating. Therefore, one of the easiest ways to obtain formulas for the multiple integrals approximate calculation is repeated approximation of simple integrals’ numerical integration formulas.

Before going on to describe the essence of the quadrature formulas’ re-use method, we’ll first consider the quadrature formulas for single integral’s values calculating. In the computational mathematics foundations [1-6] the essence of the Simpson formula for approximate calculation of values subordinate to the law of
the curve \( y = f(x) \) of a parabola, passing through \( M_0(x_0, y_0), \ M_1(x_1, y_1), \ M_2(x_2, y_2) \) three points, and is given by

\[
\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \\
\text{(1)}
\]

Where \( x_2 - x_0 = 2h \).

Now we’ll give the general Simpson’s formulas. Let \( n = 2m, \ m \in \mathbb{N}, \ y_i = f(x_i) \ (i = 0, n) \) be the value for \( y = f(x) \) function with \( h = \frac{b-a}{2n} \) steps.

Applying Simpson’s formula to each twice doubled \([x_{2i-2}, x_{2i}]\) \((i = 1, m)\) length of the interval 2h, we obtain

\[
\int_{a}^{b} f(x)dx = \frac{h}{3} \sum_{i=0}^{2m} [f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})] \\
\text{(2)}
\]

We’ll illustrate repeated Simpson’s quadrature formulas on the example of double integral of

\[
J_2 = \iint_{(D)} f(x, y)dxdy
\]

type calculating.

Using this approach, we first described how to obtain the Simpson integration formula for the double integrals approximate calculation over rectangular and curved areas, according to [5].

First, suppose that the domain of integration is a rectangle (see figure):

\[
D = \{a \leq x \leq A; b \leq y \leq B\},
\]

with sides parallel to the axes.

Each of the intervals \([a, A]\) and \([b, B]\) we split in two points

\[
x_0 = a, \ x_1 = a+h_x, \ x_2 = a+2h_x,
\]

and accordingly:

\[
y_0 = b, \ y_1 = b+h_y, \ y_2 = y+2h_y,
\]

where \(h_x = (A-a)/2, \ h_y = (B-b)/2\).

All in all, we get nine points \((x_i, y_j), \ (i,j=0,1,2)\).

We have \( J_2 = \iint_{(D)} f(x, y)dxdy = \int_{a}^{b} \int_{y}^{y_2} f(x, y)dydx \).

\[
J_2 = \int_{a}^{b} f(x, y)dydx = \int_{a}^{b} \int_{y}^{y_2} f(x, y)dydx
\]

\[
J_2 = \int_{a}^{b} \int_{y}^{y_2} f(x, y)dxdy
\]
2. Algorithm description

Theorem 1. Let the \( z = f(x, y) \) function be defined and continuous in a bounded two-dimensional domain of integration \( \Omega \). Then the cubature formula, obtained by repeated application of Simpson, has the form

\[
\int \int f(x, y) dx \; dy = \frac{h_x h_y}{9} \sum_{i=0}^{2m} \sum_{j=0}^{2n} \lambda_{ij} f_{ij};
\]

where \( \Lambda = \begin{bmatrix} 1 & 4 & 2 & 4 & 2 & 4 & 4 & 1 \\ 4 & 16 & 8 & 16 & 8 & 16 & 16 & 4 \\ 2 & 8 & 4 & 8 & 4 & 8 & 4 & 4 \\ 4 & 16 & 8 & 16 & 8 & 16 & 16 & 4 \\ 1 & 4 & 2 & 4 & 2 & 4 & 4 & 1 \end{bmatrix} \). (4)

Proof. In the theorem proof, we’ll use the Simpson’s quadrature formula’s re-use method.

According to (1), calculating the inner integral by Simpson quadrature formula, we find

\[
J_2 = \int \int f(x, y) dx \; dy = \int_a^b dx \frac{h_y}{3} [f(x, y_0) + 4f(x, y_1) + f(x, y_2)] =
\]

\[
= \frac{h_y}{3} \left[ \int_a^b f(x, y_0) dx + 4 \int_a^b f(x, y_1) dx + \int_a^b f(x, y_2) dx \right].
\]

Again applying to each integral Simpson formula, we obtain

\[
J_2 = \int \int f(x, y) dx \; dy = \frac{h_x h_y}{9} \left\{ [f(x_0, y_0) + 4f(x_1, y_0) + f(x_2, y_0)] + 
+ 4[f(x_0, y_1) + 4f(x_1, y_1) + f(x_2, y_1)] + [f(x_0, y_2) + 4f(x_1, y_2) + f(x_2, y_2)] \right\}
\]

Or

\[
J_2 = \int \int f(x, y) dx \; dy = \frac{h_x h_y}{9} \left\{ [f(x_0, y_0) + f(x_2, y_0) + f(x_0, y_2) + f(x_2, y_2)] + 
+ 4[f(x_1, y_0) + f(x_0, y_1) + f(x_2, y_1) + f(x_1, y_2)] + 16f(x_1, y_1) \right\}. \quad (5)
\]

Equation (5) is called the Simpson’s cubature formula.

Consequently,

\[
J_2 = \int \int f(x, y) dx \; dy = \frac{h_x h_y}{9} (\sigma_0 + 4\sigma_1 + 16\sigma_2),
\]
where \(-\sigma_0 = f(x_0, y_0) + f(x_2, y_0) + f(x_0, y_2) + f(x_2, y_2)\) - is the sum of the values of the integrand at the vertices of the rectangle \(D\), 
\(\sigma_1 = f(x_1, y_0) + f(x_0, y_1) + f(x_2, y_1) + f(x_1, y_2)\) - the sum of the values at the midpoints of the rectangle \(D\), 
\(\sigma_2 = f(x_1, y_1)\) - the value of the function \(f(x, y)\) in the center of the rectangle \(D\).

If the dimensions of the rectangle \(D = \{a \leq x \leq A; b \leq y \leq B\}\) are large, then to increase the accuracy of the cubature formula (5), \(D\) is divided into a system of rectangles, each of which uses the Simpson’s cubature formula.

We assume that the sides of the rectangle \(D\) are divided, respectively, by \(n\) and \(m\) equal parts; the result is a relatively large network of \(nm\) rectangles.

Each of these boxes, in turn, is divided into four equal parts. The top of this last small network of rectangles we’ll take for cubature formula’s \(M_{ij}\) nodes.

Let 
\[hx = (A-a) / (2n),\]
\[hy = (B-b) / (2m).\]

Then the network has the following coordinates:
\[x_i = a + ih_x, i=0,1,2,...,2n,\]
\[y_j = b + jh_y, j=0,1,2,...,2m.\]

To reduce, introduce the notation 
\[f(x_i, y_j) = f_{ij}.\]

Applying (5) to each of the rectangles larger network, we obtain

\[
J_2 = \iint_{(D)} f(x, y) dx dy = \frac{h_x h_y}{9} \sum_{i=0}^{2n} \sum_{j=0}^{2m} \left[ f_{2i,2j} + f_{2i+1,2j} + f_{2i,2j+1} + f_{2i+1,2j+1} + 4(f_{2i+1,2j+1} + f_{2i+1,2j+1} + f_{2i,2j+1} + f_{2i+1,2j+1}) \right].
\]  

(6)

Expanding the sum of (6), we write them as
\[
f_{00} + 4f_{01} + 2f_{02} + 4f_{03} + ... + 2f_{0,2m-2} + 4f_{0,2m-1} + f_{02m} +
+ 4f_{10} + 16f_{11} + 8f_{12} + 16f_{13} + ... + 8f_{1,2m-2} + 16f_{1,2m-1} + 4f_{1,2m} +
+ 2f_{20} + 8f_{21} + 4f_{22} + 8f_{23} + ... + 4f_{2,2m-2} + 8f_{2,2m-1} + 2f_{2,2m} +
\]

(7)

Hence we result similar terms and taking into account (7), we obtain the matrix of the(4) form. The result is the formula (3) for the approximate calculation of the double integral, obtained with the method of repeated application of Simpson’s quadrature formula. The theorem is proved.

Curved region of integration.

If the \(\Omega\) domain of integration is curved, then we construct a rectangle \(D\) such that \(\sigma \subset D\) and its sides are parallel to the axes.
Let’s consider the auxiliary function

\[ f^*(x, y) = \begin{cases} f(x, y), & \text{если } (x, y) \in \Omega; \\ 0, & \text{если } (x, y) \in D - \Omega. \end{cases} \]

In this case, we obviously have

\[ J_2 = \int \int f(x, y)dx dy = \int \int f^*(x, y)dx dy. \]

The last integral can be approximately calculated by the total cubature formula (3).

Note that the equation of the boundary, with the help of which the area of \( \Omega \) point is determined, is built by V.L. Rvachev’s R-functions [5-6].

Next, using the above-mentioned approach to calculate the concise integrals’ value, we determine the values of three and four-fold integrals.

Let’s consider the triple integral.

Theorem 2. Let the function \( u = f(x, y, z) \) be defined and continuous in a \( \Omega \) three-dimensional region of integration. Then the cubature formula obtained by repeated application of Simpson, takes the form

\[
\begin{aligned}
\iiint_{(D)} f(x, y, z)dx dy dz &= \frac{h_x h_y h_z}{27} \left\{ f_{2(2),2k} + \ldots + 64 f_{2(1,2k+1,2k+1)} \right\}, \\
J_3 &= \iiint_{a b c} f(x, y, z)dx dy dz = \frac{h_x h_y h_z}{27} \sum_{i=0}^{2n_x} \sum_{j=0}^{2n_y} \sum_{k=0}^{2n_z} \left\{ f_{2(i),2j,2k} + f_{2(i+2),2j,2k} + f_{2(i,2j+2),2k} + f_{2(i,2j+2),2k+2} \right\} + \\
&+ \left\{ f_{2(i,2j+1,2k+1),2k} + f_{2(i,2j,2k+1,2k+1)} + f_{2(i,2j+1,2k+1,2k+1)} \right\} + \\
&+ \left\{ f_{2(i+1,2j+1,2k+1,2k+1)} + f_{2(i+1,2j+1,2k+1,2k+1)} + f_{2(i+1,2j+1,2k+1,2k+1)} \right\} + \\
&+ 16 \left\{ f_{2(i+1,2j+1,2k+1,2k+1)} + f_{2(i+1,2j+1,2k+1,2k+1)} + f_{2(i,2j+1,2k+1,2k+1)} \right\} + \\
&+ 64 \left\{ f_{2(i+1,2j+1,2k+1,2k+1)} \right\}. 
\end{aligned}
\]

Proof. In the proof of the theorem we’ll use the method of Simpson’s quadrature formula three times:

\[
\begin{aligned}
J_3 &= \iiint_{a b c} f(x, y, z)dx dy dz = \int \int \int f(x, y, z)dx dy dz = \frac{h_x h_y h_z}{3} \left\{ f(x, y, z_0) + 4 f(x, y, z_1) + f(x, y, z_2) \right\} = \\
&= \frac{h_x h_y}{3} \int \int f(x, y, z_0)dx dy + \frac{4 h_x h_y}{3} \int \int f(x, y, z_1)dx dy + \frac{h_x h_y}{3} \int \int f(x, y, z_2)dx dy. 
\end{aligned}
\]
Dividing the \([a, A] \times [b, B] \times [c, C]\) three-dimensional domain into, respectively, \(n_1, n_2, n_3\) equal parts, we introduce the notation. It identifies the integration steps:

\[
x_i = a + ih_x, \quad i = 0, 2n_1,
\]
\[
y_j = b + jh_y, \quad j = 0, 2n_2,
\]
\[
z_k = c + kh_z, \quad k = 0, 2n_3.
\]

It is known that

\[
\int_{a}^{b} \int_{c}^{d} f(x, y) dx \, dy = \frac{h_x h_y}{27} \left\{ f(x_0, y_0) + f(x_0, y_1) + f(x_0, y_2) + f(x_1, y_0) + f(x_1, y_2) + f(x_2, y_0) + f(x_2, y_1) + f(x_2, y_2) \right\} + 4 \left[ f(x_0, y_0) + f(x_0, y_1) + f(x_0, y_2) + f(x_1, y_0) + f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_0) + f(x_2, y_1) + f(x_2, y_2) \right] + 16 f(x_1, y_1).
\]

Based on this formula, we have:

\[
J_3 = \int_{a}^{b} \int_{c}^{d} \int_{e}^{f} f(x, y, z) dx \, dy \, dz = \frac{h_x h_y h_z}{27} \left\{ f(x_0, y_0, z_0) + f(x_0, y_1, z_0) + f(x_0, y_2, z_0) + f(x_1, y_0, z_0) + f(x_1, y_1, z_0) + f(x_1, y_2, z_0) + f(x_2, y_0, z_0) + f(x_2, y_1, z_0) + f(x_2, y_2, z_0) \right\} + 4 \left[ f(x_0, y_0, z_0) + f(x_0, y_1, z_0) + f(x_0, y_2, z_0) + f(x_1, y_0, z_0) + f(x_1, y_1, z_0) + f(x_1, y_2, z_0) + f(x_2, y_0, z_0) + f(x_2, y_1, z_0) + f(x_2, y_2, z_0) \right] + 16 \left[ f(x_0, y_0, z_0) + f(x_0, y_1, z_0) + f(x_0, y_2, z_0) + f(x_1, y_0, z_0) + f(x_1, y_1, z_0) + f(x_1, y_2, z_0) + f(x_2, y_0, z_0) + f(x_2, y_1, z_0) + f(x_2, y_2, z_0) \right] + 64 f(x_1, y_1, z_1).
\]

Introducing the \(f(x_i, y_j, z_k) = f_{ijk}\) notation and making the replacement of \(x, y, z\) variables, respectively, we’ll write (9) in a compact form as (8).

Now, let’s consider the four-time integral.

Theorem 3. Let the function \(v = f(x, y, z, u)\) be defined and continuous in a \(\Omega\) four-dimensional integration. Then the cubature formula, obtained by the 4 times use of Simpson quadrature formula has the form
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\[
\int_{a}^{b} \int_{c}^{d} \int_{e}^{f} f(x, y, z, u) dx dy dz du = \frac{h_{1}h_{2}h_{3}h_{4}}{81} \sum_{i=0}^{2n} \sum_{j=0}^{2n} \sum_{k=0}^{2n} \sum_{l=0}^{2n} \\
(\sum \sum \sum \int_{a}^{b} \int_{c}^{d} \int_{e}^{f} f(x, y, z, u) dx dy dz du = \frac{n!}{(n-4)!} \int_{a}^{b} \int_{c}^{d} \int_{e}^{f} f(x, y, z, u) dx dy dz du)
\]

\[= \left\{ \left[ f_{1}(x, y, z, u) + f_{1}(x, y, z, u) + f_{1}(x, y, z, u) + f_{1}(x, y, z, u) \right] + \\
+ \left[ f_{2}(x, y, z, u) + f_{2}(x, y, z, u) + f_{2}(x, y, z, u) + f_{2}(x, y, z, u) \right] + \\
+ \left[ f_{3}(x, y, z, u) + f_{3}(x, y, z, u) + f_{3}(x, y, z, u) + f_{3}(x, y, z, u) \right] + \\
+ \left[ f_{4}(x, y, z, u) + f_{4}(x, y, z, u) + f_{4}(x, y, z, u) + f_{4}(x, y, z, u) \right] \right\}
\]

Proof. We’ll use the results, obtained by the triple results’ calculating:

\[J_{a} = \int_{a}^{b} \int_{c}^{d} \int_{e}^{f} f(x, y, z, u) dx dy dz du = \int_{a}^{b} \int_{c}^{d} \int_{e}^{f} dx dy dz = \int_{a}^{b} \int_{c}^{d} \int_{e}^{f} f(x, y, z, u) du = \\
= \int_{a}^{b} \int_{c}^{d} \int_{e}^{f} dx dy dz \frac{h_{1} h_{2} h_{3} h_{4}}{3} [f(x, y, z, u_{0}) + 4f(x, y, z, u_{1}) + f(x, y, z, u_{2})]
\]

Expanding the sum, we have...
Using the result, obtained in the proof of the theorem (2), i.e., using (9), we obtain

\[ J_4 = \int \int \int f(x,y,z) dx dy dz = \int \int \int \int \int \int \int \int f(x,y,z) dx dy dz = \]

\[ = \frac{h_x h_y h_z}{81} \left( f(x_0, y_0, z_0, u_0) + f(x_2, y_0, z_0, u_0) + f(x_0, y_2, z_0, u_0) + f(x_2, y_2, z_0, u_0) \right) + \]

\[ + \left[ f(x_0, y_0, z_2, u_0) + f(x_2, y_0, z_2, u_0) + f(x_0, y_2, z_2, u_0) + f(x_2, y_2, z_2, u_0) \right] + \]

\[ + \left[ f(x_0, y_0, z_0, u_2) + f(x_2, y_0, z_0, u_2) + f(x_0, y_2, z_0, u_2) + f(x_2, y_2, z_0, u_2) \right] + \]

\[ + \left[ f(x_0, y_0, z_1, u_2) + f(x_2, y_0, z_1, u_2) + f(x_0, y_2, z_1, u_2) + f(x_2, y_2, z_1, u_2) \right] + \]

\[ + \left[ f(x_0, y_0, z_2, u_1) + f(x_2, y_0, z_2, u_1) + f(x_0, y_2, z_2, u_1) + f(x_2, y_2, z_2, u_1) \right] + \]

\[ + \left[ f(x_0, y_0, z_0, u_1) + f(x_2, y_0, z_0, u_1) + f(x_0, y_2, z_0, u_1) + f(x_2, y_2, z_0, u_1) \right] + \]

\[ + \left[ f(x_0, y_0, z_1, u_1) + f(x_2, y_0, z_1, u_1) + f(x_0, y_2, z_1, u_1) + f(x_2, y_2, z_1, u_1) \right] + \]

\[ + \left[ f(x_0, y_0, z_2, u_2) + f(x_2, y_0, z_2, u_2) + f(x_0, y_2, z_2, u_2) + f(x_2, y_2, z_2, u_2) \right] + \]

\[ + 16 \left[ \int \int f(x_0, y_0, z_1, u_0) + f(x_2, y_1, z_1, u_0) + f(x_1, y_2, z_1, u_0) + f(x_1, y_2, z_1, u_0) \right] + \]

\[ + \left[ f(x_0, y_0, z_0, u_0) + f(x_2, y_0, z_2, u_0) \right] + \]

\[ + \left[ f(x_0, y_0, z_1, u_0) + f(x_2, y_0, z_1, u_0) + f(x_0, y_2, z_1, u_0) + f(x_2, y_2, z_1, u_0) \right] + \]

\[ + \left[ f(x_0, y_0, z_2, u_0) + f(x_2, y_0, z_2, u_0) + f(x_0, y_2, z_2, u_0) + f(x_2, y_2, z_2, u_0) \right] + \]

\[ + \left[ f(x_0, y_0, z_0, u_1) + f(x_2, y_0, z_0, u_1) + f(x_0, y_2, z_0, u_1) + f(x_2, y_2, z_0, u_1) \right] + \]

\[ + \left[ f(x_0, y_0, z_1, u_1) + f(x_2, y_0, z_1, u_1) + f(x_0, y_2, z_1, u_1) + f(x_2, y_2, z_1, u_1) \right] + \]

\[ + \left[ f(x_0, y_0, z_2, u_1) + f(x_2, y_0, z_2, u_1) + f(x_0, y_2, z_2, u_1) + f(x_2, y_2, z_2, u_1) \right] + \]

\[ + \left[ f(x_0, y_0, z_0, u_2) + f(x_2, y_0, z_0, u_2) + f(x_0, y_2, z_0, u_2) + f(x_2, y_2, z_0, u_2) \right] + \]

\[ + \left[ f(x_0, y_0, z_1, u_2) + f(x_2, y_0, z_1, u_2) + f(x_0, y_2, z_1, u_2) + f(x_2, y_2, z_1, u_2) \right] + \]

\[ + \left( f(x_0, y_0, z_2, u_2) + f(x_2, y_0, z_2, u_2) + f(x_0, y_2, z_2, u_2) + f(x_2, y_2, z_2, u_2) \right) + \]

\[ + 256 f(x_0, y_0, z_1, u_1). \]

Dividing the \([a,A] \times [b,B] \times [c,C] \times [d,D]\) four-dimensional area’s sides by \(n_1, n_2, n_3\) and \(n_4\) equal parts, respectively, and introducing the notation

\[ h_x = (A - a) / (2n_1), \quad h_y = (B - b) / (2n_2), \]

\[ h_z = (C - c) / (2n_3), \quad h_u = (D - d) / (2n_4), \]

we’ll define the integration steps:
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We’ll introduce the notation

\[ f(x_i, y_j, z_k, u_l) = f_{ijkl} \]

Applying the formula (2) to each of the rectangles of large network, we obtain, in this case, (10), and it gives the proof.

The given cubature formulas (5), (9), (11) are bulky. They should be written in a compact form. To do this, let’s first consider the Simpson quadrature formula:

\[ \int_{x_0}^{x_i} f(x)dx = \frac{h_x}{3}[f(x_0) + 4f(x_1) + f(x_2)]. \]  

This integral can be rewritten as

\[ \int_{x_0}^{x_i} f(x)dx = \frac{h_x}{3} \sum_{j=0}^{2}(C_j^2) f(x_j) = \frac{h_x}{3} \sum_{j=0}^{2}(C_j^2) f_j. \]

We’ll see the equivalence of the right sides of (12) and (13), opening the right side of (13). Using (13), the ratio can be rewritten as

\[ \int \int f(x, y)dx dy = \frac{h_x h_y}{9} \left\{ f(x_0, y_0) + f(x_2, y_0) + f(x_0, y_2) + f(x_2, y_2) \right\} + 
\]

\[ + 4\left\{ f(x_1, y_0) + f(x_2, y_1) + f(x_1, y_2) \right\} + 16f(x_1, y_1) \]  

\[ = \frac{h_x h_y}{9} \sum_{j=0}^{2} \sum_{j=0}^{2} (C_j^2) f(x_j, y_j) = \frac{h_x h_y}{9} \sum_{j=0}^{2} \sum_{j=0}^{2} (C_j^2) (C_j^2) f_{ij}. \]

Similarly, on the basis of (13), (9) and (11), respectively, take the form

\[ \int \int \int f(x, y, z)dx dy dz = \frac{h_x h_y h_z}{27} \sum_{i=0}^{2} \sum_{j=0}^{2} \sum_{k=0}^{2} (C_i^2 C_j^2 C_k^2) f(x_i, y_j, z_k), \]

\[ \int \int \int f(x, y, z, u)dx dy dz du = \frac{h_x h_y h_z h_u}{27} \sum_{i=0}^{2} \sum_{j=0}^{2} \sum_{k=0}^{2} \sum_{l=0}^{2} (C_i^2 C_j^2 C_k^2 C_l^2) f(x_i, y_j, z_k, u_l). \]

Now, on the basis of formulas (13) - (16), we’ll derive cubature formulas for n-fold integrals.

Before presenting the cubature formulas for n-fold integral, we’ll rewrite the formulas (1), (6), (8), (10) in a compact form, taking into account (12) - (16):

\[ \int_{a}^{b} f(x)dx = \frac{h_x^{2n}}{3} \sum_{h_i=0}^{2n} \sum_{j_i=0}^{2n} (C_{ij}^h)^2 f(x_{h_i}, j_i) = \frac{h_x^{2n}}{3} \sum_{h_i=0}^{2n} \sum_{j_i=0}^{2n} (C_{ij}^h)^2 f_{h_i j_i}, \]
\[
\int f(x, y)dx = \frac{h_x h_y}{9} \sum_{i=0}^{2n_x} \sum_{j=0}^{2n_y} \sum_{i=0}^{2} \sum_{j=0}^{2} \left( C_{ij}^x \right)^2 \left( C_{ij}^y \right)^2 f_{2i+1, 2j+1},
\]

(18)

\[
\int f(x, y, z)dydz = \frac{h_x h_y h_z}{27} \sum_{i=0}^{2n_x} \sum_{j=0}^{2n_y} \sum_{k=0}^{2n_z} \sum_{i=0}^{2} \sum_{j=0}^{2} \sum_{k=0}^{2} \left( C_{ij}^x \right)^2 \left( C_{ij}^y \right)^2 \left( C_{ij}^z \right)^2 f_{2i+1, 2j+1, 2k+1},
\]

(19)

\[
\int f(x, y, z, u)dydzdu = \frac{h_x h_y h_z h_u}{81} \sum_{i=0}^{2n_x} \sum_{j=0}^{2n_y} \sum_{k=0}^{2n_z} \sum_{l=0}^{2n_u} \left( C_{ij}^x \right)^2 \left( C_{ij}^y \right)^2 \left( C_{ij}^z \right)^2 \left( C_{ij}^u \right)^2 f_{2i+1, 2j+1, 2k+1, 2l+1}.
\]

(20)

The correctness of (17) - (20) can be determined by the amount’s disclosure.

On the basis of the above formulas we’ll state the theorem to calculate the n-multiple integrals.

**Theorem 4.** Let the \( f(x, y, \ldots, x_n) \) function be defined and continuous on a bounded n-dimensional domain of integration. Then the cubature formula to calculate the values of n-fold integrals, obtained by repeated application of the Simpson quadrature formula has the form

\[
\int f(x, y, \ldots, x_n)dx_n \approx \prod_{i=1}^{n} \left( \frac{h_i}{3} \right) \sum_{i=0}^{2n_i} \sum_{j=0}^{2n_i} \left( C_i^x \right)^2 \left( C_i^y \right)^2 \left( C_i^z \right)^2 \ldots \left( C_i^{x_n} \right)^2 f_{2i+1, 2j+1, \ldots, 2n_i+1}.
\]

(21)

Proof. In the proof we’ll use induction.

The validity of (21) for \( n = 1, 2, 3, 4 \) follows from the above formulas for the calculation of one-, two-, three- and four-fold integrals, which are described above.

Now, assuming the validity of (19) with \( n = n \), we’ll show it with \( n = n + 1 \):

\[
\int f(x_1, x_2, \ldots, x_n, x_{n+1})dx_n \approx \prod_{i=1}^{n} \left( \frac{h_i}{3} \right) \sum_{i=0}^{2n_i} \sum_{j=0}^{2n_i} \left( C_i^x \right)^2 \left( C_i^y \right)^2 \left( C_i^z \right)^2 \ldots \left( C_i^{x_n} \right)^2 \left( C_i^{x_{n+1}} \right)^2 f_{2i+1, 2j+1, \ldots, 2n_i+1, 2n_{n+1}+1} = \frac{h_{n+1}}{3} \prod_{i=1}^{n} \left( \frac{h_i}{3} \right) \sum_{i=0}^{2n_i} \sum_{j=0}^{2n_i} \left( C_i^x \right)^2 \left( C_i^y \right)^2 \left( C_i^z \right)^2 \ldots \left( C_i^{x_{n+1}} \right)^2 f_{2i+1, 2j+1, \ldots, 2n_i+1, 2n_{n+1}+1, 2n_{n+1}+1}.
\]

Thus the theorem is proved.
Now, let’s consider how to calculate the values of multiple integrals by quadrature formulas ‘ repeated application by Gauss’ method. Computation starts with single integral

\[ Y = \int_a^b f(x) \, dx \]

By introducing the following change of variable,

\[ x = \frac{b + a}{2} + \frac{b - a}{2} t, \]

we get

\[ \int_a^b f(x) \, dx = \frac{b - a}{2} \sum_{i=1}^{n} A_i f(x_i), \]

(22)

where

\[ x_i = \frac{b + a}{2} + \frac{b - a}{2} t_i \quad (i = 1, 2, ..., n), \]

(23)

\[ t_i \sim \text{Zeros of the Legendre polynomials } P_n(t), \text{ie} \]

\[ P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [x(-1)^n] = \Theta_n \quad n = 0, 1, 2, ... \]

(24)

and they are called the nodes of Gauss’ n-points. Here \( A_i \) – Gauss’ weights.

At present, the tables for the values \( A_i \) and \( x_i \) (i = 0, 1, ..., n) are drawn up. Among the most well-known and often used is a simple Kronrod table [3].

Now, we’ll consider the double integral and apply the method of repeated application of a quadrature formula:

\[ J_2 \approx \int_a^b \int_a^b f(x, y) \, dx \, dy \approx \int_a^b \left[ \frac{B - b}{2} \sum_{j=1}^{n} B_j f(x_j, y) \right] \, dx \approx \]

(25)

\[ \approx \frac{A - a}{2} \frac{B - b}{2} \sum_{j=1}^{n} A_j \sum_{j=1}^{n} B_j f(x_j, y_j) = \frac{A - a}{2} \frac{B - b}{2} \sum_{j=1}^{n} \sum_{j=1}^{n} A_j B_j f(x_j, y_j). \]

In the last formula for each \( x_i \) and \( A_j \) the values \( B_j, y_j \) and \( f(x_j, y_j) \) are calculated.

In the same way, for three-and four-time integrals, we obtain the following cubature formula:

\[ J_3 \approx \int_a^b \int_a^c \int_a^c f(x, y, z) \, dx \, dy \, dz \approx \frac{A - a}{2} \frac{B - b}{2} \frac{C - c}{2} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} A_j B_j C_k f(x_j, y_j, z_k), \]

(26)
Based on the result of one-, two-, three-, and four-time integrals’ evaluating, we’ll give the formula for approximate calculation of \( n \)-fold integral, on the base of \( n \)-point Gauss by the quadrature formulas’ reapplying.

Theorem 5. Let the \( f(x_1, x_2, \ldots, x_n) \) function be defined and continuous in a \( n \)-dimensional bounded domain of \( \Omega_n \) integration. Then the cubature formula, obtained by repeated application of Gauss’ quadrature formulas has the form

\[
Y \equiv \int_{a_1}^{A_1} \cdots \int_{a_n}^{A_n} f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n \approx \prod_{i=1}^{n} \left( \frac{A_i - a_i}{2} \right) \sum_{k=1}^{a_i} \sum_{j=1}^{A_i} \sum_{l=1}^{a_j} \sum_{m=1}^{A_j} \sum_{k=1}^{a_k} \sum_{l=1}^{A_k} \sum_{m=1}^{a_m} A_i A_j A_k \ldots \prod_{i=1}^{n} \frac{A_i - a_i}{2} f(x_1, x_2, \ldots, x_n, \ldots, x_n) \right)
\]

Proof. Let’s use induction.

The validity of (22) for \( n = 1, 2, 3, 4 \) follows from the formulas for one-, two-, three- and four-fold integrals’ calculation.

Next, assuming the validity of (22) for \( n = k \), we’ll show the validity for \( n = k + 1 \) also

\[
Y = \int_{a_{k+1}}^{A_{k+1}} \int_{a_{k+1}}^{A_{k+1}} \cdots \int_{a_{k+1}}^{A_{k+1}} f(x_1, x_2, \ldots, x_{k+1}) dx_1 dx_2 \ldots dx_k dx_{k+1} =
\]

\[
= \int_{a_{k+1}}^{A_{k+1}} \prod_{i=1}^{k+1} \left( \frac{A_i - a_i}{2} \right) \sum_{k=1}^{a_i} \sum_{j=1}^{A_i} \sum_{l=1}^{a_j} \sum_{m=1}^{A_j} \sum_{k=1}^{a_k} \sum_{l=1}^{A_k} \sum_{m=1}^{a_m} A_i A_j A_k \ldots \prod_{i=1}^{n} \frac{A_i - a_i}{2} f(x_1, x_2, \ldots, x_k, x_{k+1}) dx_{k+1} =
\]

\[
= \frac{A_{k+1} - a_{k+1}}{2} \prod_{i=1}^{k+1} \left( \frac{A_i - a_i}{2} \right) \sum_{k=1}^{a_i} \sum_{j=1}^{A_i} \sum_{l=1}^{a_j} \sum_{m=1}^{A_j} \sum_{k=1}^{a_k} \sum_{l=1}^{A_k} \sum_{m=1}^{a_m} A_i A_j A_k \ldots \prod_{i=1}^{n} \frac{A_i - a_i}{2} f(x_1, x_2, \ldots, x_k, x_{k+1}) =
\]

So it is ,what was required to prove.

3. Conclusions

Thus we have constructed the multi-dimensional cubature formulas to approximate the value of multiple integrals by repeated application of Simpson and Gauss quadrature formulas, which can be easily implemented on a computer.
REFERENCES


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