New Method for Solving Coupled Schrödinger KdV Equation

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Abstract

In this study, Fourier transform and Variational Iteration Method (FTVIM) for finding analytical solutions of the coupled Schrödinger–KdV equation is considered. The available analytical solutions of the coupled Schrödinger–KdV equation obtained by multiple traveling wave
method are compared with FTVIM to examine the accuracy of the method. Comparing the methodology with some other known techniques shows that the present approach is effective and powerful. Moreover FTVIM indicates that the amount of computational work is much less than the computational work required for both the previous VIM and the modified VIM.

**Keywords:** Fourier transform and Variational Iteration Method (FTVIM); the coupled Schrödinger–KdV equation.

### 1 Introduction

The solutions of the nonlinear evolution equations play an important role in the field of nonlinear wave phenomena. The exact solutions facilitate the verification of numerical methods when they exist. In the last few decades, substantial progress has been made on researches in this area and it continues with this direction. Multiple traveling wave solutions of nonlinear evolution equations such as the coupled Schrödinger–KdV equation [1, 2] have been successfully applied to get exact solutions by Fan [1]. The model equation for the coupled Schrödinger–KdV equation can be presented in the following form

\[ iU_t = U_{xx} + U\varphi, \quad w_t + 6ww_x + w_{xxx} = (|U|^2)_x \]  \hspace{1cm} (1)

Where \( i^2 = -1 \). The FTVIM was first proposed by S.S.Nourazar [2] and The FTVIM is useful to obtain exact and approximate solutions of linear and nonlinear differential equations. In this method, general Lagrange multipliers are introduced to construct correction functionals for the problems. The multipliers can be identified optimally via the variational theory. There is no need of linearization or discretization, and large computational work and round-off errors is avoided. It has been used to solve effectively, easily and accurately a large class of nonlinear problems with approximation [3]. The main goal of the present study is to find the analytical solutions of the coupled Schrödinger–KdV equation by the variational iteration method.

### 2 Basic Idea of Fourier Transform Variational Iteration Method (FTVIM)

The general forms of time-dependent one-dimensional nonlinear partial differential equations are considered for illustrating the idea (FTVIM) method; the presented idea is considered as follows

\[ E(u(x,t)) = \varphi(x,t), \quad x \geq 0, \quad t \geq 0 \]  \hspace{1cm} (2)

The operator \( E \) can, generally speaking, be divided into two parts, linear and nonlinear, as:

\[ L(u(x,t)) + N(u(x,t)) = \varphi(x,t) \]  \hspace{1cm} (3)

\[ u(x,0) = f(x) \]  \hspace{1cm} (4)
And the boundary conditions are presented in Eq. 5:
\[ u(0,t) = g_0(t), u_x(0,t) = g_1(t) \]  
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\[ u(0,t) = g_0(t), u_x(0,t) = g_1(t) \]  
\[ u(0,t) = g_0(t), u_x(0,t) = g_1(t) \]

Applying Fourier Transform to Eq. 3 and 4 we obtain:
\[ F\{L(u(x,t))\} + F\{N(u(x,t))\} = F\{\varphi(x,t)\}, F\{u(x,t)\} = F\{f(x)\} \]
\[ F\{L(u(x,t))\} + F\{N(u(x,t))\} = F\{\varphi(x,t)\}, F\{u(x,t)\} = F\{f(x)\} \]
\[ F\{L(u(x,t))\} + F\{N(u(x,t))\} = F\{\varphi(x,t)\}, F\{u(x,t)\} = F\{f(x)\} \]
\[ F\{L(u(x,t))\} + F\{N(u(x,t))\} = F\{\varphi(x,t)\}, F\{u(x,t)\} = F\{f(x)\} \]

Where, \( F \) indicates the Fourier transform. We develop a correction functional to eq. 6 as follow:
\[ F\{u(x,t)\}_n = F\{u(x,t)\}_n + \int_0^\xi \hat{\lambda}(\xi)(F\{L(u(x,\xi))\}_n + F\{N(\tilde{u}(x,\xi))\}_n - F\{\varphi(x,\xi)\})d\xi \]
\[ F\{u(x,t)\}_n = F\{u(x,t)\}_n + \int_0^\xi \hat{\lambda}(\xi)(F\{L(u(x,\xi))\}_n + F\{N(\tilde{u}(x,\xi))\}_n - F\{\varphi(x,\xi)\})d\xi \]
\[ F\{u(x,t)\}_n = F\{u(x,t)\}_n + \int_0^\xi \hat{\lambda}(\xi)(F\{L(u(x,\xi))\}_n + F\{N(\tilde{u}(x,\xi))\}_n - F\{\varphi(x,\xi)\})d\xi \]
\[ F\{u(x,t)\}_n = F\{u(x,t)\}_n + \int_0^\xi \hat{\lambda}(\xi)(F\{L(u(x,\xi))\}_n + F\{N(\tilde{u}(x,\xi))\}_n - F\{\varphi(x,\xi)\})d\xi \]

Where, \( \tilde{u} \) is a restricted variation and the first variation of it, \( \delta \tilde{u}(x,\xi) \), vanishes. Taking the first variation from both sides of Eq. 7 we get the following:
\[ \delta u_n(x,t) = F\{\delta u_n(x,t)\}_n + \int_0^\xi \hat{\lambda}(\xi)F\{L(\delta u_n(x,\xi))\}_n d\xi \]
\[ \delta u_n(x,t) = F\{\delta u_n(x,t)\}_n + \int_0^\xi \hat{\lambda}(\xi)F\{L(\delta u_n(x,\xi))\}_n d\xi \]
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If we apply Fourier transform in eq. 8 we will have:
\[ F\{u_n(x,t)\} = i\omega F\{u(x,t) - u(0,t)\} \]
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And so on. The main step in the Variational iteration method is to calculate the optimal value of \( \hat{\lambda}(\xi) \) in Eq.8. Making use of Eq. 9 the optimal value of \( \hat{\lambda}(\xi) \) using eq. 7 is obtained as follow:
\[ \hat{u}_n(x,t) = \hat{u}_n(x,t) + \int_0^\xi \lambda(\xi)(F\{L(\hat{u}_n(x,\xi))\}_n - \phi(\hat{u}_n(x,\xi)) + \frac{\partial}{\partial \xi^2}\tilde{u}_n(\omega,\xi) + \frac{\partial}{\partial \xi} \tilde{u}_n(\omega,\xi) + \tilde{u}_n(\omega,\xi) d\xi \]
\[ \hat{u}_n(x,t) = \hat{u}_n(x,t) + \int_0^\xi \lambda(\xi)(F\{L(\hat{u}_n(x,\xi))\}_n - \phi(\hat{u}_n(x,\xi)) + \frac{\partial}{\partial \xi^2}\tilde{u}_n(\omega,\xi) + \frac{\partial}{\partial \xi} \tilde{u}_n(\omega,\xi) + \tilde{u}_n(\omega,\xi) d\xi \]
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Superscripts denote the Fourier transform. Taking the first variation from Eq.10 we get the following:
\[ \hat{u}_n(x,t) = \delta \hat{u}_n(x,t) + \delta \left( \int_0^\xi \hat{\lambda}(\xi)(F\{L(\hat{u}_n(x,\xi))\}_n - \phi(\hat{u}_n(x,\xi)) + \frac{\partial}{\partial \xi^2}\tilde{u}_n(\omega,\xi) + \frac{\partial}{\partial \xi} \tilde{u}_n(\omega,\xi) + \tilde{u}_n(\omega,\xi) d\xi \right) \]
\[ \hat{u}_n(x,t) = \delta \hat{u}_n(x,t) + \delta \left( \int_0^\xi \hat{\lambda}(\xi)(F\{L(\hat{u}_n(x,\xi))\}_n - \phi(\hat{u}_n(x,\xi)) + \frac{\partial}{\partial \xi^2}\tilde{u}_n(\omega,\xi) + \frac{\partial}{\partial \xi} \tilde{u}_n(\omega,\xi) + \tilde{u}_n(\omega,\xi) d\xi \right) \]
\[ \hat{u}_n(x,t) = \delta \hat{u}_n(x,t) + \delta \left( \int_0^\xi \hat{\lambda}(\xi)(F\{L(\hat{u}_n(x,\xi))\}_n - \phi(\hat{u}_n(x,\xi)) + \frac{\partial}{\partial \xi^2}\tilde{u}_n(\omega,\xi) + \frac{\partial}{\partial \xi} \tilde{u}_n(\omega,\xi) + \tilde{u}_n(\omega,\xi) d\xi \right) \]
\[ \hat{u}_n(x,t) = \delta \hat{u}_n(x,t) + \delta \left( \int_0^\xi \hat{\lambda}(\xi)(F\{L(\hat{u}_n(x,\xi))\}_n - \phi(\hat{u}_n(x,\xi)) + \frac{\partial}{\partial \xi^2}\tilde{u}_n(\omega,\xi) + \frac{\partial}{\partial \xi} \tilde{u}_n(\omega,\xi) + \tilde{u}_n(\omega,\xi) d\xi \right) \]

Taking integration by parts, eq.11 can be written as:
\[ \delta \hat{u}_n(x,t) = \delta \hat{u}_n(x,t) + \lambda(t) \delta \hat{u}_n(x,t) - \lambda(0) \delta \hat{u}_n(x,t) - \frac{\partial}{\partial t} \lambda(t) \delta \hat{u}_n(x,t) + \frac{\partial}{\partial t} \lambda(t) \mid_{t=0}^\xi \delta \hat{u}_n(x,t) + \lambda(0) \frac{\partial}{\partial t} \delta \hat{u}_n(x,t) \mid_{t=0}^\xi \]
\[ \delta \hat{u}_n(x,t) = \delta \hat{u}_n(x,t) + \lambda(t) \delta \hat{u}_n(x,t) - \lambda(0) \delta \hat{u}_n(x,t) - \frac{\partial}{\partial t} \lambda(t) \delta \hat{u}_n(x,t) + \frac{\partial}{\partial t} \lambda(t) \mid_{t=0}^\xi \delta \hat{u}_n(x,t) + \lambda(0) \frac{\partial}{\partial t} \delta \hat{u}_n(x,t) \mid_{t=0}^\xi \]

Equation 12 yields the stationary condition as:
\[ \frac{1}{2} + \lambda (t) - \frac{\partial}{\partial t} \lambda (t) \right|_{t=0} = 0 \]
\[ \lambda (\xi) \right|_{\xi=0} = 0 \]
\[ \left[ (- \omega^2 + i \omega) \lambda (\xi) - \frac{\partial}{\partial \xi} \lambda (\xi) + \frac{\partial^2}{\partial \xi^2} \lambda (\xi) \right] \right|_{\xi=0} = 0 \]  

Equation 12 in turn gives the optimal value of \( \lambda \) which, we call this optimal value \( \lambda_{opt} \), substituting this back into eq.10 and by adding the nonlinear operator from eq.7 gives the iteration formula as:

\[ \hat{u}_{n+1}(\omega,t) = \hat{u}_n(\omega,t) + \int_0^L \lambda_{opt} \left( F \left[ N (u(x,\xi)) \right] \right)_n - \phi (\omega, \xi) + \frac{\partial}{\partial \xi} \hat{u}_n (\omega, \xi) + \frac{\partial^2}{\partial \xi^2} \hat{u}_n (\omega, \xi) \right| d\xi \]  

Using eq.14 the successive approximations \( \hat{u}_{n+1}(\omega,t) \), \( n \geq 0 \) can be obtained and the exact solution to the nonlinear partial differential equation is then equal to \( u = \lim_{n \to \infty} u_n \). Equation 14 introduces the recursive relations as follows:

\[ \hat{u}_0 (\omega,t) = \hat{u}(\omega,0) \]  
\[ \hat{u}_1 (\omega,t) = \hat{u}_0 (\omega,t) + \int_0^L \lambda_{opt} \left( F \left[ N (u(x,\xi)) \right] \right)_0 - \phi (\omega, \xi) + \frac{\partial}{\partial \xi} \hat{u}_0 (\omega, \xi) + \frac{\partial^2}{\partial \xi^2} \hat{u}_0 (\omega, \xi) \right| d\xi \]  
\[ \hat{u}_2 (\omega,t) = \hat{u}_1 (\omega,t) + \int_0^L \lambda_{opt} \left( F \left[ N (u(x,\xi)) \right] \right)_1 - \phi (\omega, \xi) + \frac{\partial}{\partial \xi} \hat{u}_1 (\omega, \xi) + \frac{\partial^2}{\partial \xi^2} \hat{u}_1 (\omega, \xi) \right| d\xi \]  

And so on. Using the Maple package we solve Eqs. 15-17 and then applying the inverse Fourier transform, that will define \( u_0, u_1, u_2, ..., u_n \) and the solution is then equal to \( u = \lim_{n \to \infty} u_n \).

### 3 Applying FTVM for Solving Coupled Schrödinger KdV Equation

In this section, two different solutions of the coupled Schrödinger–KdV equation [3] will be examined by using the FTVM. By using \( U = u + iv \), one can separate Eq. (1) into real and imaginary parts. Therefore, one can get a \((1 + 1)\)-dimensional tripled system in the following form

\[ u_t - v_{xx} - vw = 0, v_t + u_{xx} + uw = 0, w_t + 6ww_x + w_{xxx} - 2uu_x - 2vv_x = 0 \]  

In order to obtain FTVM solution of the Eq. (18), we apply Fourier transform on these equations

\[ F \{ u_t - v_{xx} - vw \} = 0, F \{ v_t + u_{xx} + uw \} = 0, F \{ w_t + 6ww_x + w_{xxx} - 2uu_x - 2vv_x \} = 0 \]  

By applying Fourier transform on second, third, fourth, fifth, and sixth term in left side, the third equation of Eqs. 19 we obtained:
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\[ F\{uu, t\} = \frac{i\omega}{2} F\{u^2\} - \frac{1}{2} \{u(0, t)\}^2, \quad F\{w_{uu}, t\} = -i \omega^3 w(t) - w''(0, t) - i \omega v'(0, t) - \omega^3 w(0, t) \]  
\[ (20-a,b) \]

Now we can write

\[ \frac{\partial}{\partial \alpha} \hat{u}(\alpha t) + \hat{v}(\alpha t) + i \alpha \mu(\alpha t) + v_1(\alpha t) - \hat{w} = 0 \]
\[ \frac{\partial}{\partial \alpha} \hat{v}(\alpha t) - i \alpha \mu(\alpha t) - u(\alpha t) + i \alpha \hat{w} = 0 \]
\[ \frac{\partial}{\partial \alpha} \hat{w}(\alpha t) + 3 \alpha \hat{w}(\alpha t) - 3 \mu(\alpha t) + i \alpha \hat{u}(\alpha t) - w'(0, t) - i \alpha v'(0, t) - \omega v(0, t) - i \alpha \hat{v}(\alpha t) + \mu(\alpha t) + \mu(\alpha t) + \nu(\alpha t) = 0 \]
\[ (21-a,b,c) \]

We construct a correction functional as:

\[ \hat{u}_{sa} = \hat{u} + \int_{0}^{t} \hat{A}(\xi) \left( \frac{\partial}{\partial \alpha} \hat{u} + \hat{v} + i \alpha \mu(\alpha t) + v_1(\alpha t) - \hat{w} \right) d\xi = 0 \]
\[ \hat{v}_{sa} = \hat{v} + \int_{0}^{t} \hat{A}(\xi) \left( \frac{\partial}{\partial \alpha} \hat{v} - \hat{u} - i \alpha \mu(\alpha t) - u(\alpha t) + i \alpha \hat{w} \right) d\xi = 0 \]
\[ \hat{w}_{sa} = \hat{w} + \int_{0}^{t} \hat{A}(\xi) \left( \frac{\partial}{\partial \alpha} \hat{w} + 3 \hat{w}(\alpha t) - 3 \mu(\alpha t) + i \alpha \hat{u}(\alpha t) - \hat{v}'(0, t) - i \alpha \hat{v}(\alpha t) - \omega \hat{v}(0, t) - i \alpha \hat{v}(\alpha t) + \mu(\alpha t) + \nu(\alpha t) \right) \right) d\xi = 0 \]
\[ (22-a,b,c) \]

Now to examine the FTVIM for the coupled Schrödinger–KdV equation, Jacobi doubly periodic wave solution \[ [1] \] is studied in the following sections.

### 4 Jacobi periodic solution to coupled Schrödinger–KdV equation

The Jacobi exact solutions are given \[ [3] \] by

\[ u = a_1 \sqrt{\frac{2c_2}{b_2 (2 - m^2)}} \cos \left( \sqrt{\frac{c_2}{2 - m^2}} \xi \right) \cos \theta, \quad v = a_1 \sqrt{\frac{2c_2}{b_2 (2 - m^2)}} \sin \left( \sqrt{\frac{c_2}{2 - m^2}} \xi \right) \sin \theta \]
\[ w = \frac{1}{4} \left( 2p + 4c_2 + \frac{a_1^2}{b_2} \right) - \frac{2c_2}{(2 - m^2)^2} \cos \left( \sqrt{\frac{c_2}{2 - m^2}} \xi \right) \]  
\[ (23-a,b,c) \]

Where \( a_1, b_2, p, c_2 \) are arbitrary constants and \( \xi = x + 2pt \),\( \theta = px + \frac{1}{4} (2p + 2p^2 - 2c_2 - \frac{a_1^2}{b_2})y \). For simplicity, \( a_1 = b_2 = p = c_2 = 1 \) are used in this analysis. Then, Eqs. (23-a,b,c) take the following form for \( m = 0 \):

\[ u = \cos(\frac{x + t}{4}), \quad v = \sin(\frac{x + t}{4}), \quad w = \frac{3}{4} \]
\[ (24-a,b,c) \]

Now we obtain from above equations initial conditions as:

\[ u(x,0) = \cos x, \quad v(x,0) = \sin x, \quad w(x,0) = \frac{3}{4} \]
\[ (25-a,b,c) \]
The boundary conditions are:

\[ u(0,t) = \cos(t/4), \quad v(0,t) = \sin(t/4), \quad w(0,t) = 3/4 \]  

(26-a,b,c)

Now we start with Fourier transform of initial conditions

\[ \hat{u}(\omega,0) = \frac{i\omega}{1 - \omega^2}, \quad \hat{v}(\omega,0) = \frac{1}{1 - \omega^2}, \quad \hat{w}(\omega,0) = -\frac{3i}{4\omega} \]  

(27-a,b,c)

Now if we substitute boundary conditions (26-a, b, c) and Fourier transform of initial conditions (27-a, b, c) in correction functional (22-a, b, c) and taking Integrating by parts we get

\[ \lambda_1(\xi) = \lambda_2(\xi) = \lambda_3(\xi) = -1 \]

Assuming \( \hat{u}_0(\omega,t) = \hat{u}(\omega,0), \quad \hat{v}_0(\omega,t) = \hat{v}(\omega,0), \quad \text{and} \quad \hat{w}_0(\omega,t) = \hat{w}(\omega,0) \)

And substituting for the value of \( \lambda_1(\xi) = \lambda_2(\xi) = \lambda_3(\xi) = -1 \), using Eq. (22-a, b, c) the successive approximations \( \hat{u}_{n+1}(\omega,t), \hat{v}_{n+1}(\omega,t), \hat{w}_{n+1}(\omega,t) \) Are obtained as follows:

\[ \hat{u}_{n+1} = \hat{u}_{n} + \int_{0}^{\xi} \lambda_1(\xi)[\frac{\partial}{\partial t}\hat{u}_{n} + \omega^2\hat{v}_{n} + i\omega\sin(t/4) + \cos(t/4) - v\hat{v}_{n}]d\xi = 0 \]  

(28-a,b,c)

\[ \hat{v}_{n+1} = \hat{v}_{n} + \int_{0}^{\xi} \lambda_2(\xi)[\frac{\partial}{\partial t}\hat{v}_{n} - \omega^2\hat{u}_{n} - i\omega\cos(t/4) + \sin(t/4) + u\hat{u}_{n}]d\xi = 0 \]

\[ \hat{w}_{n+1} = \hat{w}_{n} + \int_{0}^{\xi} \lambda_3(\xi)[\frac{\partial}{\partial t}\hat{w}_{n} + 3i\omega\hat{v}_{n} - 3[\sin(t/4)]^2 - i\omega\hat{u}_{n} - \frac{3}{4}\omega^2 - \omega^2\hat{u}_{n}^2 + [\cos(t/4)]^2 - i\omega\hat{u}_{n}^2 + [\sin(t/4)]^2]d\xi = 0 \]

By integrating the recursive approximate equations by parts, Eq. (28-a, b, c), and using the maple package for getting inverse Fourier transform we obtain:

\[ u_1(x,t) = \cos x - \frac{1}{4}t\sin x, \quad v_1(x,t) = \sin x + \frac{1}{4}t\cos x, \quad w_1(x,t) = \frac{3}{4} \]  

(29-a,b,c)

\[ u_2(x,t) = \cos x - \frac{1}{4}t\sin x - \frac{1}{32}t^2\cos x, \quad v_2(x,t) = \sin x + \frac{1}{4}t\cos x - \frac{1}{32}t^2\sin x, \quad w_2(x,t) = \frac{3}{4} \]  

(30-a,b,c)

And so on. Fig. 1 and 2 show the comparison between exact and FTVIM solutions of \( u(x,t) \) and \( v(x,t) \) and we can see that the results of FTVIM are very good.
Fig. 1. The comparison between exact and FTVIM solutions of $u(x,t)$ and $v(x,t)$

5 Conclusions

To put the issue into perspective, FTVIM is a powerful tool which is capable of handling linear and nonlinear partial differential equations. In this study the Fourier Transform Variational Method was used for the coupled Schrödinger-KdV equation with initial and boundary conditions. The FTVIM reduces the volume of calculations and has great advantageous which is, applying boundary condition in addition to initial condition.

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