Bounds for the Identric Mean in Terms of One-Parameter Mean

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Abstract

For $p \in \mathbb{R}$ the $p$-th one-parameter mean $J_p(a, b)$ of two positive

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1This work was supported by the Natural Science Foundation of China (Nos. 11071069 and 11171307) and the Natural Science Foundation of Zhejiang Province (Nos.LY13H070004 and LY13A010004).

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numbers \(a\) and \(b\) with \(a \neq b\) is defined by
\[
J_p(a, b) = \begin{cases} 
\frac{p(a^{p+1} - b^{p+1})}{(p+1)(a^p - b^p)}, & p \neq 0, -1, \\
\frac{a-b}{a^p - b^p}, & p = 0, \\
\frac{a-b}{a^p - b^p}, & p = -1.
\end{cases}
\] (1.1)

In this article, we answer the question: What are the greatest value \(\alpha\) and the least value \(\beta\), such that the double inequality
\[J_\alpha(a, b) < I(a, b) < J_\beta(a, b)\]
holds for all \(a, b > 0\) with \(a \neq b\)? Here \(I(a, b) = \frac{1}{e} \left( \frac{a}{b} \right)^{\frac{1}{a-b}}\) denotes the identric mean of \(a\) and \(b\).

**Mathematics Subject Classification:** 26E60, 26D20

**Keywords:** one-parameter mean, identric mean, power mean

1. **Introduction**

For \(p \in \mathbb{R}\) the \(p\)-th one-parameter mean \(J_p(a, b)\) of two positive numbers \(a\) and \(b\) with \(a \neq b\) is defined by
\[
J_p(a, b) = \begin{cases} 
\frac{p(a^{p+1} - b^{p+1})}{(p+1)(a^p - b^p)}, & p \neq 0, -1, \\
\frac{a-b}{a^p - b^p}, & p = 0, \\
\frac{a-b}{a^p - b^p}, & p = -1.
\end{cases}
\]

It is well known that \(J_p(a, b)\) is continuous and strictly increasing with respect to \(p \in \mathbb{R}\) for fixed \(a, b > 0\) with \(a \neq b\).

Let \(A(a, b) = \frac{a+b}{2}\), \(I(a, b) = \frac{1}{e} \left( \frac{a}{b} \right)^{\frac{1}{a-b}}\), \(L(a, b) = \frac{a-b}{\log a - \log b}\), \(G(a, b) = \sqrt{ab}\), and \(H(a, b) = \frac{2ab}{a+b}\) be the arithmetic, identric, logarithmic, geometric, and harmonic means of two positive numbers \(a\) and \(b\) with \(a \neq b\), respectively. Then
\[
\min\{a, b\} < H(a, b) = J_{-2}(a, b) < G(a, b) = J_{-\frac{1}{2}}(a, b)
< L(a, b) = J_0(a, b) < I(a, b) < A(a, b) = J_1(a, b) < \max\{a, b\}.
\]

For \(r \in \mathbb{R}\) the \(r\)-th power mean \(M_r(a, b)\) of two positive numbers \(a\) and \(b\) is defined by
\[
M_r(a, b) = \begin{cases} 
\left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}}, & r \neq 0, \\
\sqrt{ab}, & r = 0.
\end{cases}
\]
Recently, the bivariate means have attracted the attention of many researchers. In particular, many remarkable inequalities can be found in the literature [1-32].

In [33], Lin established the following sharp double inequality

\[ M_0(a, b) < L(a, b) < M_{\frac{3}{2}}(a, b) \]

for all \( a, b > 0 \) with \( a \neq b \).

The following best possible inequality between identric and power mean can be found in [34]:

\[ M_{\frac{3}{2}}(a, b) < I(a, b) < M_{\log_2}(a, b) \]

for all \( a, b > 0 \) with \( a \neq b \).

The following sharp bounds for \( \sqrt{L(a, b)}I(a, b) \) and \( \frac{1}{2}(L(a, b) + I(a, b)) \) in terms of power mean are proved in [35, 36]:

**Theorem A.** For all \( a, b > 0 \) with \( a \neq b \), we have

\[ M_0(a, b) < \sqrt{L(a, b)}I(a, b) < M_{\frac{3}{2}}(a, b), \]

\[ M_{\frac{3}{2}}(a, b) < \frac{1}{2}(L(a, b) + I(a, b)) < M_{\log_2}(a, b), \]

and the given parameters are the best possible.

In [35, 37-39], the authors obtained the bounds of \( L(a, b) \), \( I(a, b) \) and \( \frac{1}{2}(L(a, b) + I(a, b)) \) in terms of \( A(a, b) \) and \( G(a, b) \) as follows:

**Theorem B.** For all positive real numbers \( a \) and \( b \) with \( a \neq b \), we have

\[ L(a, b) < \frac{1}{3}A(a, b) + \frac{2}{3}G(a, b), \]

\[ \frac{2}{3}A(a, b) + \frac{1}{3}G(a, b) < I(a, b) \]

and

\[ \sqrt{A(a, b)G(a, b)} < \sqrt{L(a, b)}I(a, b) < \frac{1}{2}(L(a, b) + I(a, b)) < \frac{1}{2}(A(a, b) + G(a, b)). \]

The following Theorems C was established by Alzer and Qiu in [36].

**Theorem C.** The double inequalities

\[ \alpha A(a, b) + (1 - \alpha)G(a, b) < I(a, b) < \beta A(a, b) + (1 - \beta)G(a, b) \]
holds for all positive real numbers \( a \) and \( b \) with \( a \neq b \) if and only if \( \alpha \leq \frac{2}{3} \) and \( \beta \geq 2e = 0.73575 \ldots \).

The main purpose of this article is to answer the question: What are the greatest value \( \alpha \) and the least value \( \beta \), such that the double inequality

\[
J_\alpha(a, b) < I(a, b) < J_\beta(a, b)
\]

holds for all \( a, b > 0 \) with \( a \neq b \)?

2. Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

**Lemma 2.1.** Let \( g(t) = (t^2 - 1)(t^2 + 4t + 1) - 4t(t^2 + t + 1) \log t \), then \( g(t) > 0 \) for \( t > 1 \).

**Proof.** Simple computations lead to

\[
\begin{align*}
g(1) &= 0, \\
g'(t) &= 4(t^3 + 2t^2 - t - 2) - 4(3t^2 + 2t + 1) \log t, \\
g'(1) &= 0, \\
g''(t) &= 4(3t^2 + t - \frac{1}{t} - 3) - (24t + 8) \log t, \\
g''(1) &= 0, \\
g'''(t) &= 24(t - \log t) - \frac{8}{t} + \frac{4}{t^2} - 20, \\
g'''(1) &= 0, \\
g^{(4)}(t) &= \frac{8}{t^3}(3t^2 + 1)(t - 1) > 0
\end{align*}
\]

for \( t > 1 \).

Therefore, Lemma 2.1 follows from (2.1)-(2.5).

The following Lemma 2.2 can be easily proved via direct computations.

**Lemma 2.2.** If \( p = \frac{1}{e-1} = 0.581977 \ldots \), then

1. \( 2p^3 + 5p^2 + p - 2 > 0 \);
2. \( 22p^4 + 45p^3 + 80p^2 + 13p - 44 > 0 \);
3. \( 22p^5 + 49p^4 + 30p^3 + 53p^2 + 2p - 48 < 0 \).

**Lemma 2.3.** Let \( G(t) = (t - 1)[-t^{2p} - pt^{p+1} + 2(p + 1)t^p - pt^{p-1} - 1] + (t^p - 1)(t^{p+1} - 1) \log t \). If \( p = \frac{1}{e-1} = 0.581977 \ldots \), then there exists \( \lambda \in (1, +\infty) \), such that \( G(t) > 0 \) for \( t \in (1, \lambda) \) and \( G(t) < 0 \) for \( t \in (\lambda, +\infty) \).

**Proof.** Let \( G_1(t) = t^{1-p}G'(t) \), \( G_2(t) = t^{1-p}G''(t) \), \( G_3(t) = t^{3+p}G_2(t) \), \( G_4(t) = t^pG_3(t) \), \( G_5(t) = t^{1-p}G_4(t) \), \( G_6(t) = t^{3-p}G''(t) \). Then simple computations


yield

\[ G(1) = 0, \quad \lim_{t \to +\infty} G(t) = -\infty, \quad (2.6) \]

\[ G_1(t) = [(2p + 1)t^{p+1} - (p + 1)t - p]\log t - 2pt^{p+1} + 2pt^p - t^{1-p} + t^{-p} \\
- p(p + 2)t^2 + (3p^2 + 5p + 1)t + p(p - 1)t^{-1} - (3p^2 + 2p + 1), \]

\[ G_1(1) = 0, \quad \lim_{t \to +\infty} G_1(t) = -\infty, \quad (2.7) \]

\[ G'_1(t) = [(2p + 1)(p + 1)t^p - (p + 1)] \log t + (1 - 2p^2)t^p + 2p^2t^{p-1} \\
-(1 - p)t^{-p} - pt^{-1-p} - 2(p + 2)t - p(1 - p)t^{-2p} + (p + 1)t^{-2p} \\
- 2p^3 + 2p^2 + 4p + 1, \]

\[ G'_1(1) = 0, \quad \lim_{t \to +\infty} G'_1(t) = -\infty, \quad (2.8) \]

\[ G_2(t) = p(p + 1)(1 + 2p) \log t - 2p(p + 1)t^{1-p} - 2p^2(1 - p)t^{-p} - (p + 1)t^{-p} \\
+ pt^{-1-p} - 2p(1 - p) - 2p(1 - p)(2 + p) - p(p + 1) + 2p^2t^{2+p} \\
- 2p(p - 1)(2 + p), \]

\[ G_2(1) = 0, \quad \lim_{t \to +\infty} G_2(t) = -\infty, \quad (2.9) \]

\[ G_3(t) = 2p^2(1 - p)t^{1+p} + p(p + 1)t^2 - p(p + 1)t - p(p + 1)(1 + 2p)t^{1-p} \\
- 2p^2(1 - p)t^{2-p} - 2p(1 - p)(2 + p)t^3 + p(p + 1)(1 + 2p)t^{2+p} \\
- 2p(p - 1)(2 + p), \]

\[ G_3(1) = 0, \quad \lim_{t \to +\infty} G_3(t) = -\infty, \quad (2.10) \]

\[ G_4(t) = 2p^2(1 - p^2)t^{2p} + 2p(p + 1)t^{p+1} - p(p + 1)t^p - 6p(1 - p)(p + 2)t^{2+p} \\
+ p(p + 1)(1 + 2p)(2 + p)t^{1+2p} - 2p^2(p - 1)(p - 2)t \\
- p(1 + p)(1 - p)(1 + 2p), \]

\[ G_4(1) = 10p(p + 1)(2p - 1) > 0, \quad \lim_{t \to +\infty} G_4(t) = -\infty, \quad (2.11) \]
\[ G_5(t) = 4p^3(1 - p^2)t^p + 2p(p + 1)^2 t - p^2(p + 1) - 2p^2(p - 1)(p - 2)t^{1-p} \\
-6p(1 - p)(p + 2)^2 t^2 + p(p + 1)(1 + 2p)^2(2 + p)t^{1+p}, \]
\[ G_5(1) = 10p(2p^3 + 5p^2 + p - 2), \quad \lim_{t \to +\infty} G_5(t) = -\infty, \quad (2.12) \]
\[ G_5'(t) = 4p^4(1 - p^2)t^{p-1} + 2p(p + 1)^2 + 2p^2(p - 1)^2(p - 2)t^{-p} \\
+12p(p - 1)(p + 2)^2 t + p(p + 1)^2(1 + 2p)^2(2 + p)t^p, \]
\[ G_5'(1) = p(22p^4 + 45p^3 + 80p^2 + 13p - 44), \quad \lim_{t \to +\infty} G_5'(t) = -\infty, \quad (2.13) \]
\[ G_5''(t) = 4p^4(1 - p^2)(p - 1)t^{p-2} + 2p^3(p - 1)^2(2 - p)t^{-p-1} \\
+12p(p - 1)(p + 2)^2 + p^2(p + 1)^2(1 + 2p)^2(2 + p)t^{-p-1}, \]
\[ G_5''(1) = p(22p^5 + 49p^4 + 30p^3 + 53p^2 + 2p - 48), \quad (2.14) \]
\[ G_6(t) = 4p^4(1 - p^2)(p - 1)(p - 2) + 2p^3(p - 1)^2(p - 2)(p + 1)t^{1-2p} \\
+ p^2(p + 1)^2(1 + 2p)^2(2 + p)(p - 1)t, \]
\[ G_6(1) = p^2(p^2 - 1)(30p^3 + 7p^2 + 15p + 2) < 0, \quad (2.15) \]
\[ G_6'(t) = 2p^3(p - 1)^2(2 - p)(p + 1)(2p - 1)t^{-2p} \\
+ p^2(p + 1)^2(1 + 2p)^2(p + 2)(p - 1) \]
\[ G_6'(1) = p^2(p^2 - 1)(30p^3 + 7p^2 + 15p + 2) < 0. \quad (2.17) \]

If \( p = \frac{1}{e-1} = 0.581977 \ldots \), then (2.16) leads to that \( G_6'(t) \) is strictly decreasing in \([1, +\infty)\). From (2.17) and the monotonicity of \( G_6'(t) \) we know that \( G_6(t) \) is strictly decreasing in \([1, +\infty)\). Therefore, \( G_6'(t) \) is strictly decreasing in \([1, +\infty)\) follows from (2.15) and the monotonicity of \( G_6(t) \).

From (2.14) and Lemma 2.2(3) together with the monotonicity of \( G_5''(t) \) we clearly see that \( G_5'(t) \) is strictly decreasing in \([1, +\infty)\).

From (2.13) and Lemma 2.2(2) together with the monotonicity of \( G_5'(t) \) we know that there exists \( \lambda_1 \in (1, +\infty) \), such that \( G_5'(t) > 0 \) for \( t \in [1, \lambda_1) \) and \( G_5'(t) < 0 \) for \( t \in (\lambda_1, +\infty) \). Hence, \( G_5(t) \) is strictly increasing in \([1, \lambda_1]\) and strictly decreasing in \([\lambda_1, +\infty)\).
It follows from (2.12) and Lemma 2.2(1) together with the piecewise monotonicity of \( G_5(t) \) that there exists \( \lambda_2 \in (1, +\infty) \), such that \( G_5(t) > 0 \) for \( t \in [1, \lambda_2) \) and \( G_5(t) < 0 \) for \( t \in (\lambda_2, +\infty) \). Hence, \( G_4(t) \) is strictly increasing in \([1, \lambda_2]\) and strictly decreasing in \([\lambda_2, +\infty)\).

From (2.11) and the piecewise monotonicity of \( G_4(t) \) we clearly see that there exists \( \lambda_3 \in (1, +\infty) \), such that \( G_3(t) \) is strictly increasing in \([1, \lambda_3]\) and strictly decreasing in \([\lambda_3, +\infty)\).

It follows from (2.10) and the piecewise monotonicity of \( G_3(t) \) that there exists \( \lambda_4 \in (1, +\infty) \), such that \( G_2(t) \) is strictly increasing in \([1, \lambda_4]\) and strictly decreasing in \([\lambda_4, +\infty)\).

Making use of (2.6)-(2.9) and the similar discussions as above we know that Lemma 2.3 is true.

3. Main Result

**Theorem 3.1.** Inequality \( J_{\frac{1}{2}}(a, b) < I(a, b) < J_{\frac{1}{2}+\varepsilon}(a, b) \) holds for \( a, b > 0 \) with \( a \neq b \), and the parameters \( \frac{1}{2} \) and \( \frac{1}{2}+\varepsilon \) are the best possible.

**Proof.** Firstly, we prove that \( I(a, b) > J_{\frac{1}{2}}(a, b) \) and \( \frac{1}{2} \) is the best possible parameter. Without loss of generality, we assume that \( a > b \). Let \( t = \sqrt{\frac{a}{b}} > 1 \), then (1.1) leads to

\[
I(a, b) - J_{\frac{1}{2}}(a, b) = b \left[ \frac{1}{e} t^{\frac{2}{2^2-1}} - \frac{t^2 + t + 1}{3} \right].
\]

Let \( f(t) = \log \left( \frac{1}{e} t^{\frac{2}{2^2-1}} \right) - \log \left( \frac{t^2 + t + 1}{3} \right) \), then simple computations yield

\[
\lim_{t \to 1^+} f(t) = 0,
\]

and

\[
f'(t) = \frac{g(t)}{(t^2 - 1)^2(t^2 + t + 1)},
\]

where \( g(t) = (t^2 - 1)(t^2 + 4t + 1) - 4t(t^2 + t + 1) \log t \).

Therefore, \( I(a, b) > J_{\frac{1}{2}}(a, b) \) follows from Lemma 2.1 and (3.1)-(3.3).

For any \( \varepsilon > 0 \) and \( x > 0 \), from (1.1) one has

\[
J_{\frac{1}{2}+\varepsilon}(1+x, 1) - I(1+x, 1) = \frac{1}{2} + \varepsilon \left( \frac{1}{(1+x)^{\frac{1}{2}+\varepsilon}} - 1 \right) - \frac{1}{e} (1+x)^{\frac{1}{2}+\varepsilon} \cdot h(x),
\]

where \( h(x) \) is the best possible.

From (3.4) one has

\[
J_{\frac{1}{2}+\varepsilon}(1+x, 1) - I(1+x, 1) = \frac{1}{2} + \varepsilon \left( \frac{1}{(1+x)^{\frac{1}{2}+\varepsilon}} - 1 \right) - \frac{1}{e} (1+x)^{\frac{1}{2}+\varepsilon} \cdot h(x),
\]
where \( h(x) = \frac{1}{\varepsilon + x}[(1 + x)^{\frac{2}{3}} + \varepsilon - 1] - \frac{1}{(\varepsilon + x)^2}[(1 + x)^{\frac{1}{2}} + \varepsilon - 1] \cdot \frac{1}{\varepsilon}(1 + x)^{\frac{1}{2} + \varepsilon} \).

Letting \( x \to 0 \) and making use of Taylor expansion we get

\[
h(x) = \left[ x + \frac{\varepsilon + \varepsilon}{2} x^2 + \frac{(\frac{\varepsilon}{2} + \varepsilon)(\frac{\varepsilon}{2} + \varepsilon)}{6} x^3 + o(x^3) \right] - \left[ x + \frac{-\varepsilon + \varepsilon}{2} x^2 + \frac{(-\frac{\varepsilon}{2} + \varepsilon)(-\frac{\varepsilon}{2} + \varepsilon)}{6} x^3 + o(x^3) \right] \\
\times \left[ 1 + \frac{1}{2} x - \frac{1}{24} x^2 + o(x^2) \right] \\
= \frac{\varepsilon}{12} x^3 + o(x^3).
\]

Equations (3.4) and (3.5) imply that for any \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \), such that \( J_{\frac{1}{\varepsilon} + \varepsilon}(1 + x, 1) > I(1 + x, 1) \) for \( x \in (0, \delta) \).

Secondly, we prove that \( J_{\frac{1}{\varepsilon}}(a, b) > I(a, b) \) and \( \frac{1}{\varepsilon - 1} \) is the best possible parameter. Without loss of generality, we assume that \( a > b \). Let \( t = \frac{a}{b} > 1 \) and \( p = \frac{1}{\varepsilon - 1} \), then (1.1) leads to

\[
J_p(a, b) - I(a, b) = b \left[ \frac{p(t^{p+1} - 1)}{(p+1)(tp - 1)} - \frac{1}{e^{t^{\frac{1}{p}}}} \right].
\]

Let \( f(t) = \log \left[ \frac{p(t^{p+1} - 1)}{(p+1)(tp - 1)} \right] - \log \left( \frac{1}{e^{t^{\frac{1}{p}}}} \right) \), then simple computations yield

\[
\lim_{t \to 1^+} f(t) = 0, \quad \lim_{t \to +\infty} f(t) = 0,
\]

\[
f'(t) = \frac{G(t)}{(tp+1 - 1)(tp - 1)(t - 1)^2},
\]

where \( G(t) \) is defined as in Lemma 2.3.

Equation (3.8) and Lemma 2.3 imply that there exists \( \lambda \in (1, +\infty) \), such that \( f(t) \) is strictly increasing in \((1, \lambda)\) and strictly decreasing in \((\lambda, +\infty)\).

From (3.7) and the piecewise monotonicity of \( f(t) \) we clearly see that \( f(t) > 0 \) for \( t > 1 \). Then (3.6) leads to \( J_{\frac{1}{\varepsilon}}(a, b) > I(a, b) \).

For any \( 0 < \varepsilon < \frac{1}{\varepsilon - 1} \) and \( x > 1 \), from (1.1) one has

\[
\lim_{x \to +\infty} \frac{I(x, 1)}{J_{\frac{1}{\varepsilon - 1} - \varepsilon}(x, 1)} = \lim_{x \to +\infty} \frac{\frac{1}{\varepsilon - 1} - \frac{1}{\varepsilon} \cdot (x^{\frac{1}{\varepsilon - 1} - \varepsilon} - 1) x^{\varepsilon} - 1}{\frac{1}{\varepsilon - 1} - \varepsilon} \\
= \frac{\frac{1}{\varepsilon - 1} - \frac{1}{\varepsilon}}{\frac{1}{\varepsilon - 1} - \varepsilon} > 1.
\]

Inequality (3.9) implies that for any \( 0 < \varepsilon < \frac{1}{\varepsilon - 1} \) there exists \( X = X(\varepsilon) > 1 \), such that \( I(x, 1) > J_{\frac{1}{\varepsilon - 1} - \varepsilon}(x, 1) \) for \( x \in (X, +\infty) \).
References


Received: May 5, 2013