The Difference of $\chi^2$ over $p$–Metric Spaces Defined by Musielak

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Abstract. In this paper, we introduce the sequence spaces $\chi_{fp}^{2\alpha} (\Delta_n^m)$ and $\Lambda_{fp}^{2\alpha} (\Delta_n^m)$ defined by Musielak. We study some topological properties and prove some inclusion relations between these spaces.

Mathematics Subject Classification: 40A05, 40C05, 40D05

Keywords: analytic sequence, double sequences, $\chi^2$ space, difference sequence space, Musielak - modulus function, $p$- metric space

1. Introduction

Throughout $w, \chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, $w^2$ is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [2]. Later on, they were investigated by Hardy [3], Moricz [7], Moricz and Rhoades [8], Basarir and Solankan [1], Tripathy [11], Turkmenoglu [12], and many others.

We procure the following sets of double sequences:

$$
\mathbb{M}_u(t) := \{(x_{mn}) \in w^2 : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \},
$$

$$
\mathcal{C}_p(t) := \{(x_{mn}) \in w^2 : \lim_{m,n \to \infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \},
$$

$$
\mathcal{C}_{0p}(t) := \{(x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn}|^{t_{mn}} = 1 \},
$$

$$
\mathcal{L}_u(t) := \{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \},
$$

$$
\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \cap \mathbb{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \cap \mathbb{M}_u(t);
$$

where $t = (t_{mn})$ is the sequence of strictly positive reals $t_{mn}$ for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n \to \infty}$ denotes the limit in the Pringsheim’s sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathbb{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathbb{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [14,15] have proved that $\mathbb{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathbb{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [16] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [17] and Tripathy [11] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [20] have defined the spaces $BS, BS(t), CS_p, CS_{bp}, CS_r$ and $BV$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathbb{M}_u, \mathbb{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces.
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bas, BV, CS$bp$ and the $\beta (\vartheta) -$ duals of the spaces $CSbp$ and $CS_r$ of double series. Basar and Sever [21] have introduced the Banach space $L_q$ of double sequences corresponding to the well-known space $\ell_q$ of single sequences and examined some properties of the space $L_q$. Quite recently Subramanian and Misra [22] have studied the space $\chi^2_M (p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [6] as an extension of the definition of strongly Cesàro summable sequences. Connor [23] further extended this definition to a definition of strong $A-$ summability where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ statistical convergence. In [24] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [25]-[26], and [27] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} x_{k,\ell}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

\[(a + b)^p \leq a^p + b^p\] (1.1)

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence $(s_{mn})$ is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$.

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{m,n} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called double gai sequence if \((m+n)! |x_{mn}|^{1/m+n} \to 0\) as $m, n \to \infty$. The double gai sequences will be denoted by $\chi^2$. Let $\phi = \{all \; finite \; sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{i,j} \Xi_{ij}$ for all $m, n \in \mathbb{N}$; where $\Xi_{ij}$ denotes the double sequence whose only non zero term is $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space) $X$ is said to have AK property if $(\Xi_{mn})$ is a Schauder basis for $X$. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Let $M$ and $\Phi$ are mutually complementary modulus functions. Then, we have:

(i) For all $u, y \geq 0$,

\[uy \leq M (u) + \Phi (y) , (Young's \; inequality) [See \; [13]]\] (1.2)
(ii) For all \( u \geq 0 \),
\[
(1.3) \quad u \eta (u) = M(u) + \Phi(\eta(u)).
\]

(iii) For all \( u \geq 0 \), and \( 0 < \lambda < 1 \),
\[
(1.4) \quad M(\lambda u) \leq \lambda M(u).
\]

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to construct Orlicz sequence space
\[
\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},
\]

The space \( \ell_M \) with the norm
\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},
\]
becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p \) \((1 \leq p < \infty)\), the spaces \( \ell_M \) coincide with the classical sequence space \( \ell_p \).

A sequence \( f = (f_{mn}) \) of modulus function is called a Musielak-modulus function. A sequence \( g = (g_{mn}) \) defined by
\[
g_{mn}(v) = \sup \{ |v| u - (f_{mn})(u) : u \geq 0 \}, m, n = 1, 2, \cdots
\]
is called the complementary function of a Musielak-modulus function \( f \). For a given Musielak modulus function \( f \), the Musielak-modulus sequence space \( t_f \) and its subspace \( h_f \) are defined as follows
\[
t_f = \left\{ x \in w^2 : I_f(\|x_{mn}\|)^{1/m+n} \to 0 as m, n \to \infty \right\},
\]
\[
h_f = \left\{ x \in w^2 : I_f(\|x_{mn}\|)^{1/m+n} \to 0 as m, n \to \infty \right\},
\]

where \( I_f \) is a convex modular defined by
\[
I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}(\|x_{mn}\|)^{1/m+n}, x = (x_{mn}) \in t_f.
\]

We consider \( t_f \) equipped with the Luxemburg metric
\[
d(x, y) = \sup_{mn} \left\{ \inf \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right) \right) \leq 1 \right\}
\]

If \( X \) is a sequence space, we give the following definitions:

(i) \( X' = \) the continuous dual of \( X \);

(ii) \( X^\alpha = \{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X \} \);

(iii) \( X^\beta = \{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convergent, for each } x \in X \} \);

(iv) \( X^\gamma = \{ a = (a_{mn}) : \sup_{mn} \geq 1 \sum_{m,n=1}^{M,N} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X \} \);

(v) let \( X \) be an FK - space \( \supset \phi; \) then \( X^f = \{ f(\Phi_{mn}) : f \in X' \} \);
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\[(vi) X^\delta = \left\{ a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{forall } x \in X \right\};\]

$X^\alpha, X^\beta, X^\gamma$ are called $\alpha - \text{(or Kôthe - Toeplitz)}$ dual of $X, \beta - \text{(or generalized - Kôthe - Toeplitz)}$ dual of $X, \gamma - \text{dual of X}$ respectively. $X^\alpha$ is defined by Gupta and Kampion [13]. It is clear that $X^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\beta \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

\[Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \}\]

for $Z = c, c_0 \text{ and } \ell_\infty$, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here $c, c_0 \text{ and } \ell_\infty$ denote the classes of convergent, null and bounded scalar valued sequences respectively. The difference sequence space $bv_p$ of the classical space $\ell_p$ is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar in [20]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and $bv_p$ are Banach spaces normed by

\[\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = (\sum_{k=1}^\infty |x_k|^p)^{1/p}, (1 \leq p < \infty).\]

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

\[Z(\Delta) = \{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \}\]

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_m + x_n = (x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1})$ for all $m, n \in \mathbb{N}$. The generalized difference double notation has the following representation: $\Delta^m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{m+1n} + \Delta^{m-1} x_{m+1n+1},$ and also this generalized difference double notation has the following binomial representation:

\[\Delta^m x_{mn} = \sum_{i=0}^m \sum_{j=0}^m (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{m+i,n+j}.\]

2. Definition and Preliminaries

Let $n \in \mathbb{N}$ and $X$ be a real vector space of dimension $w$, where $n \leq w$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on $X$ satisfying the following four conditions:

(i) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,

(ii) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_1), \ldots, d_n(x_n))\|_p = |\alpha| \|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$

(iv) $d_p ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) = (d_X(x_1, x_2, \ldots, x_n)^p + d_Y(y_1, y_2, \ldots, y_n)^p)^{1/p}$ for $1 \leq p < \infty$; (or)

(v) $d ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) := \sup \{ d_X(x_1, x_2, \ldots, x_n), d_Y(y_1, y_2, \ldots, y_n) \},$

for $x_1, x_2, \ldots, x_n \in X, y_1, y_2, \ldots, y_n \in Y$ is called the $p$ product metric of the Cartesian product of $n$ metric spaces is the $p$ norm of the $n$-vector of the norms of the $n$ subspaces.

A trivial example of $p$ product metric of $n$ metric space is the $p$ norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the $p$ norm:
∥(d_1(x_1),\ldots,d_n(x_n))\|_E = \sup \left| \det\left(\begin{array}{cccc} d_{11}(x_{11}) & d_{12}(x_{12}) & \cdots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \cdots & d_{2n}(x_{1n}) \\ \vdots & \vdots & & \vdots \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \cdots & d_{nn}(x_{nn}) \end{array} \right) \right|

where x_i = (x_{i1},\ldots,x_{in}) \in \mathbb{R}^n for each i = 1,2,\ldots,n.

If every Cauchy sequence in X converges to some L ∈ X, then X is said to be complete with respect to the p- metric. Any complete p- metric space is said to be p- Banach metric space.

Let X be a linear metric space. A function w : X → \mathbb{R} is called paranorm, if
(1) w(x) ≥ 0, for all x ∈ X;
(2) w(-x) = w(x), for all x ∈ X;
(3) w(x + y) ≤ w(x) + w(y), for all x, y ∈ X;
(4) If (σ_{mn}) is a sequence of scalars with σ_{mn} → σ as m, n → ∞ and (x_{mn}) is a sequence of vectors with w(x_{mn} - x) → 0 as m, n → ∞, then w(σ_{mn}x_{mn} - σx) → 0 as m, n → ∞.

A paranorm w for which w(x) = 0 implies x = 0 is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [32], Theorem 10.4.2, p.183). The zero sequence
\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}

is denoted by θ and p = (p_{mn}) is a sequence of strictly positive real numbers. Further the sequence (p_{mn}^{-1}) will be represented by (t_{mn}).

Let f = (f_{mn}) be a Musielak-modulus function and p = (p_{mn}) be any bounded sequence of positive real numbers and let (X, q) be a seminormed space seminormed by q. In the present paper, we define the following sequence spaces:

Let us consider μ_{mn}(x) = \left[ q((m+n)!\Delta_m^n)^{1/m+n} \right]^{p_{mn}} t_{mn}

χ_{fp}^q(Δ_m^n) = \{ x = (x_{mn}) ∈ X : [f(μ_{mn}(x))] → 0, as m, n → ∞ \},

Λ_{fp}^q(Δ_m^n) = \{ x = (x_{mn}) ∈ X : sup_{mn} [f(μ_{mn}(x))] < ∞ \}.

If we take p = (p_{mn}) = 1, we have
\( \chi_{f^{p}}^{2q} (\Delta_{n}^{m}) = \{ x = (x_{mn}) \in X : [f(\mu_{mn}(x))] \to 0, \text{as} \, m, n \to \infty \} \),

\( \Lambda_{f^{p}}^{2q} (\Delta_{n}^{m}) = \{ x = (x_{mn}) \in X : sup_{mn} [f(\mu_{mn}(x))] < \infty \} \).

The following inequality will be used throughout the paper. If \( 0 \leq p_{mn} \leq sup_{mn} = K, D = max \{ 1, 2^{K-1} \} \) then

\[ |a_{mn} + b_{mn}|^{p_{mn}} \leq D \{ |a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}} \} \]

for all \( m, n \) and \( a_{mn}, b_{mn} \in \mathbb{C} \). Also \( |a|^{p_{mn}} \leq max \{ 1, |a|^{K} \} \) for all \( a \in \mathbb{C} \).

In this paper we study some topological properties of the above sequence spaces.

3. Main Results

3.1. Theorem. Let \( f = (f_{mn}) \) be a Musielak-modulus function, \( p = (p_{mn}) \) be a double analytic sequence of strictly positive real numbers, the sequence spaces \( \left[ \chi_{f^{p}}^{2q}, \|(d(x_{1}), d(x_{2}), \cdots, d(x_{n-1}))\|_{p} \right] \) and \( \left[ \Lambda_{f^{p}}^{2q}, \|(d(x_{1}), d(x_{2}), \cdots, d(x_{n-1}))\|_{p} \right] \) are linear spaces.

Proof: It is routine verification. Therefore the proof is omitted.

3.2. Theorem. Let \( f = (f_{mn}) \) be a Musielak-modulus function, \( p = (p_{mn}) \) be a double analytic sequence of strictly positive real numbers, the sequence space \( \left[ \chi_{f^{p}}^{2q}, \|(d(x_{1}), d(x_{2}), \cdots, d(x_{n-1}))\|_{p} \right] \) is a paranormed space with respect to the paranorm defined by

\( g(x) = \inf \left\{ \left[ f_{mn} \left( \|(\mu_{mn}(x), d(x_{1}), d(x_{2}), \cdots, d(x_{n-1}))\|_{p} \right) \right] \leq 1 \right\} = 0. \)

Proof: Clearly \( g(x) \geq 0 \) for \( x = (x_{mn}) \in \left[ \chi_{f^{p}}^{2q}, \|(d(x_{1}), d(x_{2}), \cdots, d(x_{n-1}))\| \right] \). Since \( f_{mn}(0) = 0 \), we get \( g(0) = 0 \).

Conversely, suppose that \( g(x) = 0 \), then

\( \inf \left\{ \left[ f_{mn} \left( \|(\mu_{mn}(x), d(x_{1}), d(x_{2}), \cdots, d(x_{n-1}))\|_{p} \right) \right] \leq 1 \right\} = 0 \)

Suppose that \( \mu_{mn}(x) \neq 0 \) for each \( m, n \in \mathbb{N} \). Then \( \|(\mu_{mn}(x), d(x_{1}), d(x_{2}), \cdots, d(x_{n-1}))\|_{p} \to \infty \). It follows that \( \left[ f_{mn} \left( \|(\mu_{mn}(x), d(x_{1}), d(x_{2}), \cdots, d(x_{n-1}))\|_{p} \right) \right]^{1/H} \to \infty \) which is a contradiction. Therefore \( \mu_{mn}(x) = 0 \). Let

\( \left[ f_{mn} \left( \|(\mu_{mn}(x), d(x_{1}), d(x_{2}), \cdots, d(x_{n-1}))\|_{p} \right) \right]^{1/H} \leq 1 \)

and

\( \left[ f_{mn} \left( \|(\mu_{mn}(y), d(x_{1}), d(x_{2}), \cdots, d(x_{n-1}))\|_{p} \right) \right]^{1/H} \leq 1 \)

Then by using Minkowski’s inequality, we have

\( \left[ f_{mn} \left( \|(\mu_{mn}(x+y), d(x_{1}), d(x_{2}), \cdots, d(x_{n-1}))\|_{p} \right) \right]^{1/H} \leq \).
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\[
\left[ f_{mn} \left( \|\mu_{mn} (x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right) \right]^{1/H} + \\
\left[ f_{mn} \left( \|\mu_{mn} (y), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right) \right]^{1/H}.
\]

So we have
\[
g(x + y) = \inf \left\{ f_{mn} \left( \|\mu_{mn} (x + y), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right) \leq 1 \right\} \leq \\
\inf \left\{ f_{mn} \left( \|\mu_{mn} (x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right) \leq 1 \right\} + \\
\inf \left\{ f_{mn} \left( \|\mu_{mn} (y), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right) \leq 1 \right\}
\]

Therefore,
\[
g(x + y) \leq g(x) + g(y).
\]

Finally, to prove that the scalar multiplication is continuous. Let \( \lambda \) be any complex number. By definition,
\[
g(\lambda x) = \inf \left\{ f_{mn} \left( \|\mu_{mn} (\lambda x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right) \leq 1 \right\}.
\]

Then
\[
g(\lambda x) = \inf \left\{ (|\lambda| t)^{q_{mn}/H} : u_{mn} \left[ f_{mn} \left( \|\mu_{mn} (\lambda x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right) \right]^{q_{mn}} \leq 1 \right\}
\]

where \( \lambda = \frac{1}{|\lambda|} \). Since \( |\lambda|^{q_{mn}} \leq \max (1, |\lambda|^{\supp_{mn}}) \), we have
\[
g(\lambda x) \leq \max (1, |\lambda|^{\supp_{mn}}) \inf \left\{ t^{q_{mn}/H} : u_{mn} \left[ f_{mn} \left( \|\mu_{mn} (\lambda x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right) \right] \leq 1 \right\}
\]

This completes the proof.

3.3. **Theorem.** (i) If the sequence \((f_{mn})\) satisfies uniform \(\Delta_2\)-condition, then
\[
\left[ \chi_{fp}^{2q}, \|\mu_{mn} (x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right) \right]^{\alpha} = \\
\left[ \chi_{g}^{2q}, \|\mu_{mn} (x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right).
\]

(ii) If the sequence \((g_{mn})\) satisfies uniform \(\Delta_2\)-condition, then
\[
\left[ \chi_{fp}^{2q}, \|\mu_{mn} (x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right) \right]^{\alpha} = \\
\left[ \chi_{fp}^{2q}, \|\mu_{mn} (x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right]
\]

**Proof:** Let the sequence \((f_{mn})\) satisfies uniform \(\Delta_2\)-condition, we get

\[
(3.1) \\
\left[ \chi_{fp}^{2q}, \|\mu_{mn} (x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right] \subset \left[ \chi_{fp}^{2q}, \|\mu_{mn} (x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right]^{\alpha}
\]

To prove the inclusion
\[
\left[ \chi_{fp}^{2q}, \|\mu_{mn} (x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right]^{\alpha} \subset \\
\left[ \chi_{fp}^{2q}, \|\mu_{mn} (x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right],
\]

let \( a \in \left[ \chi_{fp}^{2q}, \|\mu_{mn} (x), (d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \right] \). Then for all \( \{x_{mn}\} \) with \( (x_{mn}) \in \)
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\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}a_{mn}| < \infty. \]

Since the sequence $(f_{mn})$ satisfies uniform $\Delta_2$-condition, then

\[ (y_{mn}) \in \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right], \]

we get

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\varphi_{rn}y_{mn}a_{mn}| < \infty. \]

by (3.2). Thus $(\varphi_{rn}a_{mn}) \in \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right]

and hence

\[ (a_{mn}) \in \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right]. \]

This gives that

\[ \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right]^\alpha \subset \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \]

we are granted with (3.1) and (3.3)

\[ \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right]^\alpha = \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \]

(ii) Similarly, one can prove that

\[ \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right]^\alpha \subset \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \]

if the sequence $(g_{mn})$ satisfies uniform $\Delta_2$-condition.

3.4. Proposition. If $0 < p_{mn} < r_{mn} < \infty$ for each $m$ and $n$, then

\[ \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \subset \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \]

Proof: The proof is standard, so we omit it.

3.5. Proposition. (i) If $0 < \inf p_{mn} \leq p_{mn} < 1$ then

\[ \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \subset \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right]. \]

(ii) If $1 \leq p_{mn} \leq \sup p_{mn} < \infty$, then

\[ \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \subset \left[ \chi_{f_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right]. \]

Proof: The proof is standard, so we omit it.

3.6. Proposition. Let $f' = (f'_{mn})$ and $f'' = (f''_{mn})$ are sequences of Musielak functions, we have

\[ \left[ \chi_{f'_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \cap \left[ \chi_{f''_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \]

\[ \left[ \chi_{f'_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \subset \left[ \chi_{f''_{np}}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \]

Proof: The proof is easy so we omit it.
3.7. **Proposition.** For any sequence of Musielak functions \( f = (f_{mn}) \) and \( q = (q_{mn}) \) be double analytic sequence of strictly positive real numbers. Then
\[
\left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \subset \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p.
\]
**Proof:** The proof follows from Proposition 3.8.

3.8. **Proposition.** The sequence space \( \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \) is solid

**Proof:** Let \( x = (x_{mn}) \in \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \), (i.e.)
\[
\sup_{mn} \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] < \infty.
\]
Let \( (\alpha_{mn}) \) be double sequence of scalars such that \( |\alpha_{mn}| \leq 1 \) for all \( m, n \in \mathbb{N} \times \mathbb{N} \). Then we get
\[
\sup_{mn} \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(\alpha x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \leq \sup_{mn} \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] .
\]
This completes the proof.

3.9. **Proposition.** The sequence space \( \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \) is monotone

**Proof:** The proof follows from Proposition 3.8.

3.10. **Proposition.** If \( f = (f_{mn}) \) be any Musielak function. Then
\[
\left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \subset \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p^*} \]
if and only if \( \sup_{r,s} \frac{\phi_{rs}^*}{\phi_{rs}^*} \leq \frac{1}{\phi_{rs}^*} < \infty \).

**Proof:** Let \( x \in \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right] \) and \( N = \sup_{r,s} \frac{\phi_{rs}^*}{\phi_{rs}^*} \). Then we get
\[
\left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p^*} \right] = N \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p^*} \right] = 0.
\]
Thus \( x \in \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p^*} \right] \). Conversely, suppose that
\[
\left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p^*} \right] \subset \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p^*} \right]
\]
\[
\left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p^*} \right] \text{ and } x \in \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p^*} \right].
\]
Then
\[
\left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p^*} \right] < \epsilon, \text{ for every } \epsilon > 0. \text{ Suppose that } \sup_{r,s} \frac{\phi_{rs}^*}{\phi_{rs}^*} = \infty, \text{ then there exists a sequence of members } (r_{s,j,k}) \text{ such that } \lim_{j,k \to \infty} \frac{\phi_{rs}^*}{\phi_{rs}^*} = \infty. \text{ Hence, we have}
\]
\[
\left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p^*} \right] = \infty. \text{ Therefore}
\]
\[
x \notin \left[ \Lambda_{fp}^{q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_{p^*} \right] , \text{ which is a contradiction. This completes the proof.}
3.11. Proposition. If \( f = (f_{mn}) \) be any Musielak function. Then
\[
\left[ \Lambda_{fpq}^2 \| \mu_{mn} (x), (d (x_1), d (x_2), \ldots, d (x_{n-1})) \|_{p^*} \right] =
\left[ \Lambda_{fpq}^2 \| \mu_{mn} (x), (d (x_1), d (x_2), \ldots, d (x_{n-1})) \|_{p^*} \right]
\] if and only if \( \sup_{r,s} \geq 1 \frac{2 \epsilon}{m^2} < \infty \), \( \sup_{r,s} \geq 1 \frac{2 \epsilon}{m^2} > \infty \).

Proof: It is easy to prove so we omit.

3.12. Proposition. The sequence space \( \left[ \chi_{fpq}^2 \| \mu_{mn} (x), (d (x_1), d (x_2), \ldots, d (x_{n-1})) \|_{p} \right] \) is not solid

Proof: The result follows from the following example.

Example: Consider
\[
x = (x_{mn}) = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & & \vdots \\
1 & 1 & \ldots & 1 \\
\end{pmatrix}
\]
then \( \alpha_{mn} = \begin{pmatrix}
-1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \\
-1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \\
\vdots & \vdots & & \vdots \\
-1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \\
\end{pmatrix} \), for all \( m,n \in \mathbb{N} \).

Then \( \alpha_{mn} x_{mn} \notin \left[ \chi_{fpq}^2 \| \mu_{mn} (x), (d (x_1), d (x_2), \ldots, d (x_{n-1})) \|_{p} \right] \). Hence \( \left[ \chi_{fpq}^2 \| \mu_{mn} (x), (d (x_1), d (x_2), \ldots, d (x_{n-1})) \|_{p} \right] \) is not solid.

3.13. Proposition. The sequence space \( \left[ \chi_{fpq}^2 \| \mu_{mn} (x), (d (x_1), d (x_2), \ldots, d (x_{n-1})) \|_{p} \right] \) is not monotone

Proof: The proof follows from Proposition 3.12.

A sequence \( x = (x_{mn}) \) is said to be \( \varphi - \) statistically convergent or \( s_{\varphi} - \) statistically convergent to 0 if for every \( \epsilon > 0 \),
\[
\lim_{r \to \infty} \left\{ \left[ f_{mn} \left( \| \mu_{mn} (x), (d (x_1), d (x_2), \ldots, d (x_{n-1})) \|_{p} \right) \right]^{q_{mn}} \right\} \geq \epsilon = 0
\]
where the vertical bars indicates the number of elements in the enclosed set. In this case we write \( s_{\varphi} - \lim x = 0 \) or \( x_{mn} \to 0 (s_{\varphi}) \) and \( s_{\varphi} = \{ x : \exists 0 \in \mathbb{R} : s_{\varphi} - \lim x = 0 \} \).

3.14. Proposition. For any sequence of Musielak functions \( f = (f_{mn}) \) and \( p = (p_{mn}) \) be double analytic sequence of strictly positive real numbers. Then
\[
\left[ \chi_{fpq}^2 \| \mu_{mn} (x), (d (x_1), d (x_2), \ldots, d (x_{n-1})) \|_{p} \right] \subset
\left[ \chi_{fpq}^2 \| \mu_{mn} (x), (d (x_1), d (x_2), \ldots, d (x_{n-1})) \|_{p} \right]
\]
Proof: Let \( x \in \left[ x_{f^p}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right] \) and \( \epsilon > 0 \). Then
\[
\left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right) \right] \
\geq \left\{ \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right) \right] \right\} \geq \epsilon
\]
from which it follows that \( x \in \left[ s_{f^p}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right] \).

To show that \( s_{f^p}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \) strictly contain
\[
\left[ \chi_{f^p}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right].
\]
We define \( x = (x_{mn}) \) by \( (x_{mn}) = mn \) if \( rs - \sqrt{\frac{r}{s}} + mn \leq rs \) and \( (x_{mn}) = 0 \) otherwise. Then
\[
x \notin \left[ \chi_{f^p}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right]
\]
and for every \( \epsilon (0 < \epsilon \leq 1) \),
\[
\left\{ \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right) \right] \right\} = \frac{\sqrt{\frac{r}{s}} + mn}{\sqrt{\frac{r}{s}} - mn} \to 0 \text{ as } r, s \to \infty
\]
i.e. \( x \to \left( s_{f^p}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right) \), where \( \| \) denotes the greatest integer function. On the other hand,
\[
\left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right) \right] \to \infty \text{ as } r, s \to \infty
\]
i.e. \( x_{mn} \not\to 0 \left[ \chi_{f^p}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right]. \) This completes the proof.

3.15. Theorem. Suppose \( f' = \left( f'_{mn} \right) \) and \( f'' \) are Musielak-modulus functions satisfying the \( \Delta_2 \)-condition then we have the following results:
(i) If \( (p_{mn}) \in \Lambda^2 \) then \( \chi_{f^p}^{2q} (\Delta_m^n) \subseteq \chi_{f_p^{f'}}^{2q} (\Delta_m^n) \)
(ii) \( \chi_{f^p}^{2q} (\Delta_m^n) \cap \chi_{f^p}^{2q} (\Delta_m^n) \subseteq \chi_{f^p}^{2q} (\Delta_m^n) \).

Proof: If \( x = (x_{mn}) \in \chi_{f^p}^{2q} (\Delta_m^n) \) then
\[
\left[ f^'_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right) \right] \to 0 \text{ as } m, n \to \infty.
\]
Suppose
\[
y_{mn} = \left[ f^'_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right) \right] \text{ for all } m, n \in \mathbb{N}.
\]
Choose \( \delta > 0 \) be such that \( 0 < \delta < 1 \), then for \( y_{mn} \geq \delta \) we have \( y_{mn} < \frac{y_{mn}}{\delta} < 1 + \frac{y_{mn}}{\delta} \). Now \( f'' \) satisfies \( \Delta_2 \)-condition so that there exists \( J \geq 1 \) such that
\[
f''_{mn} \left( \frac{y_{mn}}{\delta} \right) f''_{mn} \left( 2 \right) = \frac{y_{mn}}{\delta} f''_{mn} \left( 2 \right).
\]
We obtain
\[
\left[ \left( f''_{mn} \circ f''_{mn} \right) \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right) \right] = \left[ f^' \left( f^'_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right) \right) \right] = \left[ f^'_{mn} \left( \|\mu_{mn}(y_{mn}), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right) \right] \to 0, \text{ as } m, n \to \infty.
\]
Similarly, we can prove the other cases.
(ii) Suppose \( x = (x_{mn}) \in \chi_{f^p}^{2q} (\Delta_m^n) \cap \chi_{f^p}^{2q} (\Delta_m^n) \), then
\[
\left[ f^'_{mn} \left( \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p \right) \right] \to 0 \text{ as } m, n \to \infty.
\]
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$$\left[f''_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p\right)\right] \to 0 \text{ as } m, n \to \infty.$$ 

The above inequality follows

$$\left(f''_{mn} \circ f'_{mn}\right) \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p\right) \leq D \left\{\left[f''_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p\right)\right] + \left[f'_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p\right)\right]\right\}.$$ 

Hence $\chi^{2q}_{f''} (\Delta^n_m) \cap \chi^{2q}_{f''} (\Delta^n_m) \subseteq \chi^{2q}_{f' + f''} (\Delta^n_m)$.

**Competing Interests:** Author have declared that no competing interests exist.

**References**


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Received: June 1, 2013