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Discrete Multi-target Linear-quadratic Control Problem and Quadratic Programming[1]

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Abstract

We consider a discrete multi-target linear-quadratic control problem. We reduce the problem into a quadratic programming over a simplex. Computing the coefficients of the cost function requires knowing the descriptions of the orthogonal project onto the vector space Z and the orthogonal complement $Z^{\perp}$ of the vector space Z.

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1 Introduction

Let $(H, <,>)$ be a Hilbert space, Z be its closed vector subspace, $h_1, \cdots, h_m$ and c be vectors in H. Consider the following optimization problem:

$$\max_{1\leq i\leq m} \|h - h_i\| \to \min, \quad h \in c + Z, \tag{1}$$

where $\|\cdot\|$ is the norm in a hilbert space H induced by the inner product $<,>$. This problem was considered in a classical optimal control theory known as a

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tracking problem or Multi-target linear quadratic problem (MTLQP). According to this notation, there are m targets and we wish to track these targets using a given linear system with a quadratic cost function. A more general version of this problem where the cost function is a general quadratic function is studied in [3] and it is called multi criteria linear quadratic control problem (MCLQP). A solution to MCLQP is also fully described [3] with the help of infinite dimensional interior-point method. A decent direction is calculated on each iteration. It is shown that the approximated solution approaches to the optimal solution. In [2], Faybusovich and Mouktonglang analyzed (1) using duality theory for infinite-dimensional second-order cone programming. They obtain a reduction of this problem to a finite dimensional second-order cone programming and apply this result to a multi-target linear-quadratic control problem on a finite time interval. Later on, [4] considers a reduction (1) to even simpler optimization problem of minimization of convex quadratic function on the $(m-1)$ dimensional simplex. Then by applying this result to the analysis of a continuous version of multi-target linear-quadratic control problem on semi-infinite time interval, they show that the coefficients of the quadratic function admitted a simple expressions in term of the original data. In this paper, we apply the result of the reduction to a discrete version of multi-target linear-quadratic control problem. We also show that the coefficients of the quadratic function admitted a simple expressions in term of the original data.

2 Reduction to a simple quadratic programming problem

The reduction to a quadratic programming problem over a simplex was completely described in full details in [4]. For the completeness of the paper, we shortly describe the reduction in this section. For more details, see [4].

Let $f_i(h) = \|h - h_i\|^2, i = 1, 2, \cdots, m$. One can easily see that (1) is equivalent to the following optimization problem

$$z \to \min, \tag{2}$$

$$f_i(h) \leq z, i = 1, 2, \cdots, m, \tag{3}$$

$$h \in c + Z. \tag{4}$$

Consider the Lagrange function

$$\begin{aligned}\mathcal{L}(\lambda_1, \cdots, \lambda_m, h, z) &= z + \sum_{i=1}^{m} \lambda_i (f_i(h) - z) \\ &= z(1 - \sum_{i=1}^{m} \lambda_i) + \sum_{i=1}^{m} \lambda_i f_i(h).\end{aligned}$$

The optimality conditions for (2) - (4) take the form

$$\lambda_i \geq 0, \;\; \lambda_i(f_i(h) - z) = 0, \; i = 0, 1, 2, \cdots, m, \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial z} = 0, \; \sum_{i=1}^{m} \lambda_i \nabla f_i(h) \in Z^{\perp}, \tag{6}$$

where $\nabla f_i(h) = 2(h - h_i), \; i = 1, 2, \cdots, m, \; Z^{\perp}$ is the orthogonal complement of Z in H. Conditions (5), (6) imply that

$$\sum_{i=0}^{m} \lambda_i = 1, \; \lambda_i \geq 0, \; i = 1, 2, \cdots, m, \tag{7}$$

$$\pi_Z(h) = \sum_{i=1}^{m} \lambda_i (\pi_Z h_i), \tag{8}$$

where $\pi_Z : H \to Z$ is the orthogonal projection onto Z. Similarly, $\pi_{Z^{\perp}} : H \to Z^{\perp}$ is the orthogonal project onto $Z^{\perp}$ of Z. Then the Lagrange dual of (2), (3),(4) takes the form

$$\varphi(\lambda_1, \lambda_2, \cdots, \lambda_m) = \min\{\mathcal{L}(\lambda_1, \cdots, \lambda_m, h, z) : h \in c + Z, \; z \in Z\}.$$

Using (7), (8), we obtain

$$\varphi(\lambda_1, \lambda_2, \cdots, \lambda_m) = \sum_{i=1}^{m} \lambda_i f_i(h(\lambda_1, \cdots, \lambda_m)) \tag{9}$$

where

$$h(\lambda_1, \cdots, \lambda_m) = \pi_{Z^{\perp}}(c) + \sum_{i=1}^{m} \lambda_i \pi_Z(h_i). \tag{10}$$

For simplicity, we let

$$h(\lambda) = \sum_{i=1}^{m} \lambda_i h_i.$$

Then

$$\begin{aligned} f_j(h(\lambda_1, \cdots, \lambda_m)) &= \|\pi_Z(h(\lambda)\|^2 + \|\pi_Z(h_j))\|^2 - 2 < \pi_Z(h(\lambda)), \pi_Z(h_j) > \\ &\quad + \|\pi_{Z^\perp}(c - h_j)\|^2. \end{aligned}$$

Hence, according to (9)

$$\begin{aligned} \varphi(\lambda_1, \cdots, \lambda_m) &= \|\pi_Z(h(\lambda))\|^2 + \sum_{j=1}^{m} \lambda_j \|\pi_Z(h_j)\|^2 \\ &\quad -2 < \pi_Z(h(\lambda)), \pi_Z(h(\lambda)) > + \sum_{j=1}^{m} \lambda_j \|\pi_{Z^\perp}(c - h_j)\|^2. \end{aligned}$$

We, hence, arrive at the following expression of φ:

$$\varphi(\lambda_1, \cdots, \lambda_m) = -\|\pi_Z(\sum_{i=1}^{m} \lambda_i h_i)\|^2 + \sum_{j=1}^{m} \lambda_j (\|\pi_Z(h_j)\|^2 + \pi_{Z^\perp}(c - h_j)\|^2). \tag{11}$$

We can simplify (11) further. Notice that

$$\|\pi_{Z^\perp}(c - h_j)\|^2 = \|\pi_{Z^\perp}(c)\|^2 + \|\pi_{Z^\perp}(h_j)\|^2 - 2 < \pi_{Z^\perp}(c), \pi_{Z^\perp}(h_j) > .$$

Consequently,

$$\begin{aligned} \varphi(\lambda_1, \cdots, \lambda_m) &= -\|\pi_Z(h(\lambda))\|^2 + \sum_{j=1}^{m} \lambda_j \|h_j\|^2 \\ &\quad -2 < \pi_{Z^\perp}(c), \pi_{Z^\perp}(h(\lambda)) > + \|\pi_{Z^\perp}(c)\|^2 \\ &= -\|h(\lambda)\|^2 + \|\pi_{Z^\perp}(h(\lambda) - c)\|^2 + \sum_{j=1}^{m} \lambda_j \|h_j\|^2 \end{aligned} \tag{12}$$

Here,

$$h(\lambda) = \sum_{i=1}^{m} \lambda_i h_i.$$

Hence, the Lagrange dual to (2), (3), (4) takes the form:

$$\varphi(\lambda_1, \cdots, \lambda_m) \to \max, \tag{13}$$

$$\sum_{i=1}^{m} \lambda_i = 1,\ \lambda_i \geq 0,\ i = 1, 2, \cdots, m. \tag{14}$$

The problem (13)-(14) is known as a quadratic programming over a simplex.

3 Discrete linear-quadratic control problem

In this section, we consider the discrete time formulation for the problem (1). For simplicity, let us introduce some useful notations. Let x denote a sequence $x = \{x_k\} \subset \mathbb{R}^n$ for $k = 0, 1, 2, \ldots, \infty$. We say that $x \in l_2^n(\mathbb{N})$ if $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$, where $\|\cdot\|$ is a norm induced by an inner product $<,>$ in $\mathbb{R}^n$. Let $(x, u) \in l_2^n(\mathbb{N}) \times l_2^m(\mathbb{N})$. We let the pair $(x, u) \in Z$, where Z is a vector subspace of the Hilbert space $l_2^n(\mathbb{N}) \times l_2^m(\mathbb{N})$. Observe now the inner product in H has the following form :

$$\langle (x, y), (u, \nu) \rangle_H = \sum_{k=0}^{\infty} \{ \langle x_k, u_k \rangle + \langle y_k, \nu_k \rangle \}$$

The vector subspace Z now takes the form :

$$Z = \{(x, u) \in H : x_{k+1} = Ax_k + Bu_k, k = 0, 1, 2, \ldots, x_0 = 0\}$$

Here A is an n by n matrix, and B is an n by m matrix.

The solution to this particular class of optimization problems can be completely described by solving system of recurrence relations and the following well known discrete algebraic Riccati equation (DARE) :

$$K = A^T K A - A^T K B (I + B^T K B)^{-1} (A^T K B)^T + I$$

We assume that this DARE has a positive definite stabilizing solution K_{st}. For sufficient conditions, see [4]. For simplicity, we let

$$\overline{I} = (I + B^T K B)$$

and we also let

$$L = B^T \overline{I}^{-1} B^T.$$

Then the next theorem describes the orthogonal complement $Z^{\perp}$ of Z.

Theorem 3.1. *The orthogonal complement $Z^{\perp}$ of Z is described as follows:*

$$Z^{\perp} = \{(A^T p_k - p_{k-1}, B^T p_k); \{p_k\} \in l_2^n(\mathbb{N}), for\ k = 0, 1, 2, \dots\}.$$

i.e. given $(\psi_k, \varphi_k) \in H$, we have

$$\begin{aligned} \psi_k &= x_k - (A^T p_k - p_{k-1}), & (15)\\ \varphi_k &= u_k - B^T p_k, & (16) \end{aligned}$$

where x_k is the solution

$$x_{k+1} = (A^T - A^T KL)^T x_k + L\rho_{k+1} + B\overline{I}^{-1}\varphi_k, \tag{17}$$

$$u_k = -\overline{I}^{-1} B^T KAx_k + \overline{I}^{-1} B^T \rho_{k+1} + \overline{I}^{-1}\varphi_k, \tag{18}$$

$$p_k = -Kx_{k+1} + \rho_{k+1}, \tag{19}$$

and ρ_k is a unique solution

$$\rho_k = (A^T + A^T KL)\rho_{k+1} - A^T KB\overline{I}^{-1}\varphi_k + \psi_k \tag{20}$$

belonging to $l_2^n(\mathbb{N})$.

In particular, $(\{x_k\}, \{u_k\}) \in Z$, $-(\{A^T p_k - p_{k-1}\}, \{B^T p_k)\} \in Z^{\perp}$ and consequently Z is a closed subspace in H with

$$\pi_Z(\{\psi_k\}, \{\varphi_k\}) = (\{x_k\}, \{u_k\}), \quad \pi_{Z^{\perp}}(\{\psi_k\}, \{\varphi_k\}) = -(\{A^T p_k - p_{k-1}\}, \{B^T p_k\}).$$

Proof: Let $(\{x_k\}, \{u_k\}) \in Z$ and $(\{A^T p_k - p_{k-1}\}, \{B^T p_k\}) \in Z^{\perp}$

$$
\begin{aligned}
&\langle(\{x_k\},\{u_k\}),(\{A^Tp_k-p_{k-1}\},\{B^Tp_k\})\rangle\\
&= \sum_{k=o}^{\infty}\big(\langle x_k, A^Tp_k-p_{k-1}\rangle+\langle u_k,B^Tp_k\rangle\big)\\
&= \langle x_0,A^Tp_0\rangle+\langle u_0,B^Tp_0\rangle+\sum_{k=1}^{\infty}\big(\langle x_k, A^Tp_k-p_{k-1}\rangle+\langle u_k,B^Tp_k\rangle\big)\\
&= \langle x_0,p_0\rangle+\langle Bu_0,p_0\rangle+\lim_{n\to\infty}\sum_{k=1}^{n}\big(\langle x_k,A^Tp_k\rangle-\langle x_k,p_{k-1}\rangle+\langle u_k,B^Tp_k\rangle\big)\\
&= \langle Ax_0+Bu_0,p_0\rangle+\lim_{n\to\infty}\sum_{k=1}^{n}\big(\langle x_k,A^Tp_k\rangle-\langle x_k,p_{k-1}\rangle+\langle u_k,B^Tp_k\rangle\big)\\
&= \langle x_1,p_0\rangle+\lim_{n\to\infty}\sum_{k=1}^{n}\big(\langle Ax_k,p_k\rangle-\langle x_k,p_{k-1}\rangle+\langle Bu_k,p_k\rangle\big)\\
&= \langle x_1,p_0\rangle+\lim_{n\to\infty}\sum_{k=1}^{n}\big(\langle Ax_k+Bu_k,p_k\rangle-\langle x_k,p_{k-1}\rangle\big)\\
&= \langle x_1,p_0\rangle+\lim_{n\to\infty}\sum_{k=1}^{n}\big(\langle x_{k+1},p_k\rangle-\langle x_k,p_{k-1}\rangle\big)\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\langle x_{n+1},p_n\rangle\\
&= 0
\end{aligned}
$$

Consider the algebraic Riccati equation [DARE]

$$K = A^TKA - A^TKLKA + I. \tag{21}$$

We are now looking for p_k in the form

$$p_{k-1} = -Kx_k + \rho_k.$$

Hence by (20),

$$
\begin{aligned}
p_{k-1} &= -Kx_k + (A^T - A^TKL)\rho_{k+1} - A^TKB\overline{I}^{-1}\varphi_k + \psi_k\\
&= -Kx_k + A^T\rho_{k+1} - A^TKL\rho_{k+1} - A^TKB\overline{I}^{-1}\varphi_k + \psi_k.
\end{aligned}
$$

Then it follows that

$$\begin{aligned}
A^T p_k - p_{k-1} &= A^T[-Kx_{k+1} + \rho_{k+1}] - p_{k-1} \\
&= -A^T K x_{k+1} + A^T \rho_{k+1} + K x_k - A^T \rho_{k+1} + A^T K L \rho_{k+1} \\
&\quad + A^T K B \overline{I}^{-1} \varphi_k - \psi_k \\
&= -A^T K[(A^T - A^T K L)^T x_k + L\rho_{k+1} + B\overline{I}^{-1}\varphi_k] + K x_k + A^T K L \rho_{k+1} \\
&\quad + A^T K B \overline{I}^{-1} \varphi_k - \psi_k \\
&= -A^T K A x_k + A^T K L A x_k - A^T K L \rho_{k+1} - A^T K B \overline{I}^{-1} \varphi_k + K x_k \\
&\quad + A^T K L \rho_{k+1} + A^T K B \overline{I}^{-1} \varphi_k - \psi_k \\
&= [-A^T k A + A^T K L K A + K] x_k - \psi_k
\end{aligned}$$

By using now the fact that K_{st} satisfies (21), we obtain :

$$A^T p_k - p_{k-1} = x_k - \psi_k$$

which is (15). By using (18), we obtain

$$\begin{aligned}
u_k - B^T p_k &= -B^T K A x_k - B^T K B u_k + B^T \rho_{k+1} + \varphi_k - B^T[-Kx_{k+1} + \rho_{k+1}] \\
&= -B^T K A x_k - B^T K B u_k + B^T \rho_{k+1} + \varphi_k + B^T K x_{k+1} - B^T \rho_{k+1} \\
&= -B^T K[A x_k + B u_k - x_{k+1}] + \varphi_k \\
&= \varphi_k
\end{aligned}$$

which is (16). Finally, by simple calculation, for x_k and u_k defined by (17), (18), we have $x_{k+1} = Ax_k + Bu_k$. Then the proof is completed.

Theorem 3.2. *Let $h = (\psi, \varphi) = (\{\psi_k\}, \{\varphi_k\}) \in H$ and $\rho \in L_2^n[0, \infty)$ is the function entering the decomposition (15) and (16) and described in (20). Then*

$$\|\pi_Z(h)\|^2 = \|C(B^T\rho + \varphi)\|^2, \tag{22}$$

$$\|\pi_{Z^\perp}(h)\|^2 = \|h\|^2 - \|C(B^T\rho + \varphi)\|^2. \tag{23}$$

Proof: Let $(\{y_k\}, \{\nu_k\}) \in Z$. and

$$\begin{aligned}
\Delta(y_k, \nu_k) &= [\nu_k + \overline{I}^{-1} B^T(KAy_k - \rho_{k+1}) - \overline{I}^{-1}\varphi_k]^T \overline{I} [\nu_k + \overline{I}^{-1} B^T(KAy_k - \rho_{k+1}) - \overline{I}^{-1}\varphi_k] \\
&= \Delta_1 + \Delta_2 + \Delta_3
\end{aligned}$$

where

$\Delta_1 = (\nu_k - \overline{I}^{-1}\varphi_k)^T \overline{I} (\nu_k - \overline{I}^{-1}\varphi_k)$,

$$\Delta_2 = (KAy_k - \rho_{k+1})^T L(KAy_k - \rho_{k+1}),$$
$$\Delta_3 = 2(KAy_k - \rho_{k+1})^T B(\nu_k - \overline{I}^{-1}\varphi_k).$$

Hence,

$$\begin{aligned}
\Delta_1 &= \nu_k^T \overline{I} \nu_k - 2\nu_k^T \varphi_k + \varphi_k^T \overline{I}^{-1} \varphi_k \\
&= \nu_k^T (I + B^T KB)\nu_k - 2\nu_k^T \varphi_k + \varphi_k^T \overline{I}^{-1} \varphi_k \\
&= \nu_k^T \nu_k + v_k^T B^T KB\nu_k - 2\nu_k^T \varphi_k + \varphi_k^T \overline{I}^{-1} \varphi_k \\
&= (\nu_k - \varphi_k)^T (\nu_k - \varphi_k) - \varphi_k^T \varphi + (y_{k+1} - Ay_k)^T K(y_{k+1} - Ay_k) + \varphi_k^T \overline{I}^{-1} \varphi_k \\
&= (\nu_k - \varphi_k)^T (\nu_k - \varphi_k) + y_{k+1}^T K y_{k+1} - 2y_{k+1}^T KAy_k + y_k^T A^T KAy_k - \varphi_k^T \varphi_k \\
&\quad + \varphi_k^T \overline{I}^{-1} \varphi_k \\
\Delta_2 &= y_k^T A^T KLKAy_k - 2y_k^T A^T KL\rho_{k+1} + \rho_{k+1}^T L\rho_{k+1} \\
\Delta_3 &= 2y_k^T A^T KB\nu_k - 2\rho_{k+1}^T B\nu_k - 2y_k^T A^T KB\overline{I}^{-1} \varphi_k + 2\rho_{k+1}^T B\overline{I}^{-1} \varphi_k \\
&= 2y_k^T A^T K(y_{k+1} - Ay_k) - 2\rho_{k+1}^T (y_{k+1} - Ay_k) - 2y_k^T A^T KB\overline{I}^{-1} \varphi_k \\
&\quad + 2\rho_{k+1}^T B\overline{I}^{-1} \varphi_k \\
&= 2y_k^T A^T Ky_{k+1} - 2y_k^T A^T KAy_k - 2\rho_{k+1}^T y_{k+1} + 2\rho_{k+1}^T Ay_k \\
&\quad - 2y_k^T A^T KB\overline{I}^{-1} \varphi_k + 2\rho_{k+1}^T B\overline{I}^{-1} \varphi_k
\end{aligned}$$

$$\begin{aligned}
\Delta(y_k, \nu_k) &= (\nu_k - \varphi_k)^T (\nu_k - \varphi_k) + y_{k+1}^T Ky_{k+1} - 2y_{k+1}^T KAy_k + y_k^T A^T KAy_k \\
&\quad - \varphi_k^T \varphi_k + \varphi_k^T \overline{I}^{-1} \varphi_k + y_k^T A^T KLKAy_k - 2y_k^T A^T KL\rho_{k+1} \\
&\quad + \rho_{k+1}^T L\rho_{k+1} + 2y_k^T A^T Ky_{k+1} - 2y_k^T A^T KAy_k - 2\rho_{k+1}^T y_{k+1} \\
&\quad + 2\rho_{k+1}^T Ay_k - 2y_k^T A^T KB\overline{I}^{-1} \varphi_k + 2\rho_{k+1}^T B\overline{I}^{-1} \varphi_k \\
&= (\nu_k - \varphi_k)^T (\nu_k - \varphi_k) - y_k^T [A^T KA - A^T KLKA] y_k \\
&\quad + 2y_k^T [A^T - A^T KL]\rho_{k+1} + y_{k+1}^T Ky_{k+1} - \varphi_k^T \varphi_k + \varphi_k^T \overline{I}^{-1} \varphi_k \\
&\quad + \rho_{k+1}^T L\rho_{k+1} - 2\rho_{k+1}^T y_{k+1} - 2y_k^T A^T KB\overline{I}^{-1} \varphi_k \\
&\quad + 2\rho_{k+1}^T B\overline{I}^{-1} \varphi_k
\end{aligned}$$

Since $A^T KA - A^T KLKA = K - I$, and

$$[A^T - A^T KL]\rho_{k+1} = \rho_k + A^T KB\overline{I}^{-1} \varphi_k - \psi_k,$$

we have

$$
\begin{aligned}
\Delta(y_k,\nu_k) &= (\nu_k-\varphi_k)^T(\nu_k-\varphi_k)+y_k^T[K-I]y_k+2y_k^T[\rho_k+A^TKB\overline{I}^{-1}\varphi_k-\psi_k] \\
&\quad +y_{k+1}^TKy_{k+1}-\varphi_k^T\varphi_k+\varphi_k^T\overline{I}^{-1}\varphi_k+\rho_{k+1}^TL\rho_{k+1}-2\rho_{k+1}^Ty_{k+1} \\
&\quad -2y_k^TA^TKB\overline{I}^{-1}\varphi_k+2\rho_{k+1}^TB\overline{I}^{-1}\varphi_k \\
&= (\nu_k-\varphi_k)^T(\nu_k-\varphi_k)+y_k^TKy_k+y_k^Ty_k+2y_k^T\rho_k-2y_k^T\psi_k \\
&\quad +y_{k+1}^TKy_{k+1}-\varphi_k^T\varphi_k+\varphi_k^T\overline{I}^{-1}\varphi_k+\rho_{k+1}^TL\rho_{k+1}-2\rho_{k+1}^Ty_{k+1} \\
&\quad +2\rho_{k+1}^TB\overline{I}^{-1}\varphi_k \\
&= (\nu_k-\varphi_k)^T(\nu_k-\varphi_k)+(y_k-\psi_k)^T(y_k-\psi_k)-\psi_k^T\psi_k-\varphi_k^T\varphi_k \\
&\quad +\varphi_k^T\overline{I}^{-1}\varphi_k+\rho_{k+1}^TB\overline{I}^{-1}B^T\rho_{k+1}+2\rho_{k+1}^TB\overline{I}^{-1}\varphi_k \\
&\quad -y_k^TKy_k+y_{k+1}^TKy_{k+1}+2\rho_k^Ty_k-2\rho_{k+1}^Ty_{k+1} \\
&= (\nu_k-\varphi_k)^T(\nu_k-\varphi_k)+(y_k-\psi_k)^T(y_k-\psi_k)-(\psi_k^T\psi_k+\varphi_k^T\varphi_k) \\
&\quad +(B^T\rho_{k+1}+\varphi_k)^T\overline{I}^{-1}(B^T\rho_{k+1}+\varphi_k) \\
&\quad -y_k^TKy_k+y_{k+1}^TKy_{k+1}+2\rho_k^Ty_k-2\rho_{k+1}^Ty_{k+1}
\end{aligned}
$$

Notice, since we fixed x_0, we can let $y_0=x_0$ and then take summation of both sides :

$$
\begin{aligned}
\sum_{k=o}^{\infty}\Delta(y_k,\nu_k) &= \sum_{k=o}^{\infty}[(\nu_k-\varphi_k)^T(\nu_k-\varphi_k)+(y_k-\psi_k)^T(y_k-\psi_k)]-\sum_{k=o}^{\infty}[\psi_k^T\psi_k+\varphi_k^T\varphi_k] \\
&\quad +\sum_{k=o}^{\infty}[(B^T\rho_{k+1}+\varphi_k)^T\overline{I}^{-1}(B^T\rho_{k+1}+\varphi_k)]-x_0^TKx_0+2\rho_0^Tx_0
\end{aligned}
$$

Using the fact that $x_0=0$

$$
\sum_{k=o}^{\infty}\Delta(y_k,\nu_k) = \|(y_k-\psi_k,\nu_k-\varphi_k)\|^2-\|(\psi_k,\varphi_k)\|^2+\|C(B^T\rho_{k+1}+\varphi_k)\|^2
$$

By the definition of $\Delta(y_k,\nu_k)$, $\Delta(x_k,u_k)=0$. Therefore

$$
0 = \|(y_k-\psi_k,\nu_k-\varphi_k)\|^2-\|(\psi_k,\varphi_k)\|^2+\|C(B^T\rho_{k+1}+\varphi_k)\|^2
$$

Since $\Delta(y_k,\nu_k)=\pi(\psi_k,\varphi_k)$

$$
\|(\psi_k,\varphi_k)\|^2 = \|C(B^T\rho_{k+1}+\varphi_k)\|^2+\|\pi_Z(\psi_k,\varphi_k)\|^2
$$

Hence,

$$
\|\pi_Z(\psi_k,\varphi_k)\|^2 = \|C(B^T\rho_{k+1}+\varphi_k)\|^2.
$$

That completes the proof.

We are now in the position to compute the coefficients for the quadratic cost function φ. Since

$$\varphi(\lambda_1, \cdots, \lambda_m) = -\|h(\lambda)\|^2 + \|\pi_{Z^\perp}(h(\lambda) - c)\|^2 + \sum_{j=1}^{m} \lambda_j \|h_j\|^2, \quad (24)$$

and $h(\lambda) = \sum_{i=1}^{m} \lambda_i h_i$, then we have

$$\begin{aligned} \|h(\lambda)\|^2 &= \|\pi_Z h(\lambda)\|^2 + \|\pi_{Z^\perp} h(\lambda)\|^2 && (25) \\ &= \|\pi_Z(\sum_{i=1}^{m} \lambda_i h_i)\|^2 + \|\pi_{Z^\perp}(\sum_{i=1}^{m} \lambda_i h_i)\|^2 && (26) \\ &= \|\sum_{i=1}^{m} \lambda_i \pi_Z h_i\|^2 + \|\sum_{i=1}^{m} \lambda_i \pi_{Z^\perp} h_i\|^2. && (27) \end{aligned}$$

Similarly, as for $\|\pi_{Z^\perp}(h(\lambda) - c)\|^2$ we can apply the theroem 2. This would allow us to express the quadratic cost function (13) in terms of given data.

4 Concluding Remarks:

In this paper, we consider the discrete multi target linear quadratic control problem. We reduce the problem into quadratice programming over a simplex. The coefficients of the cost functions can be computed by knowing the descriptions of the orthogonal project onto the vector space Z and the orthogonal complement $Z^\perp$ of the vector space Z.

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