Regular Decagon Cover for

Isoperimetric Triangles

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Abstract

A region covers a family of arcs if it contains a congruent copy of every arc in the family. In this paper, we show that the regular decagon of diameter one is the smallest regular decagon which covers the family of all triangles of perimeter two.

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1 Introduction

The maximum and the minimum of the distance between the two parallel support lines of the set $X$ are called the diameter of the set $X$ and the thickness of the set $X$, respectively. Note that the diameter of any regular decagon is the length of the diagonal. Moreover, the diameter and the thickness of any triangle are the length of the longest side and the length of the altitude to the longest side, respectively. For a region $R$ and a family $F$ of arcs, we say that $R$ covers $F$ or $R$ is a cover for $F$, if $R$ contains a congruent copy of each arc in $F$. In 2013, Sroysang [3,4,7] presented the smallest regular pentagon, the smallest regular hexagon and
the smallest regular heptagon for the family of all triangles of perimeter two. In this paper, we show that the regular decagon of diameter one is the smallest regular decagon which covers the family of all triangles of perimeter two. For related results, we refer to the references [1,3,4,8,9,10].

2 Results

Theorem The smallest regular decagon cover for the family of all triangles of perimeter two is the regular decagon of diameter one (the area is equal to \(\frac{5}{16}(3-\sqrt{5})\sqrt{5+2\sqrt{5}}\)).

Proof. Let \(R\) be the regular decagon of diameter one and let \(T\) be a triangle of perimeter two. We denote the vertices of the triangle \(T\) by \(A\), \(B\) and \(C\) where the angle \(\angle A\) is greater than or equal to the angle \(\angle B\), and the angle \(\angle B\) is greater than or equal to the angle \(\angle C\). Let \(D\) be a vertex of the regular decagon \(R\) and let \(P\) and \(Q\) be two points on the perimeter of the regular decagon \(R\) such that the distance between the vertex \(D\) and the segment \(PQ\) is equal to 0.58 and the segment \(PQ\) is parallel to a diagonal of the regular decagon \(R\) as shown in Fig. 1. We note that \(|PQ| < 0.9206\). Now, we divide the triangle \(T\) into two cases.

Case 1. The diameter of the triangle \(T\) is at most \(|PQ|\). We note on this case that the thickness of the triangle \(T\) is less than 0.282. WLOG, we can put the triangle \(T\) into the regular decagon \(R\) where the segment \(BC\) lies on the segment \(PQ\), and \(C = P\). This implies that the regular decagon \(R\) contains a congruent copy of the triangle \(T\).

Fig. 1. The segment \(PQ\) in the regular decagon \(R\)

Case 2. The diameter of the triangle \(T\) is greater than \(|PQ|\). We note on this case that the thickness of the triangle \(T\) is less than 0.282. WLOG, we can put the triangle \(T\) into the regular decagon \(R\) where the segment \(BC\) lies on a diagonal of
the regular decagon $R$ and the vertex $A$ is above than the segment $BC$ as shown in Fig. 2 or Fig. 3. Now, the vertex $A$ may be in the regular decagon $R$ or not in the regular decagon $R$.

Fig. 2. A triangle $T$ in the regular decagon $R$ where the vertex $A$ is in $R$

Fig. 3. A triangle $T$ in the regular decagon $R$ where the vertex $A$ is not in $R$

Suppose for a contradiction that the regular decagon $R$ does not contains the triangle $T$ as shown in Fig. 3. Let $G$ be the intersection of the segment $AC$ and the segment $EF$ and let $H$ be the point on the segment $BC$ such that the segment $GH$ is perpendicular to the segment $BC$ as shown in Fig. 4.
Fig. 4. A right triangle $GHC$ in the triangle $T$ and the regular decagon $R$

Let $x$ be the length of the segment $FH$. Then $0 \leq x < 0.1$ and the length of the segment $HC$ is $1 - x$. Note that the angle $HFG$ is equal to $\frac{2\pi}{5}$. Then the length of the segment $GH$ is $x \tan \left( \frac{2\pi}{5} \right)$.

Define $L(x) = 1 - x + x \tan \left( \frac{2\pi}{5} \right) + \sqrt{(1 - x)^2 + \left(x \tan \left( \frac{2\pi}{5} \right) \right)^2}$. Then $L(x)$ is the total length of the perimeter of the right triangle $GHC$. By the calculation on $L(x)$, we obtain that $L(x) \geq L(0) = 2$ as shown in Fig. 5.

Fig. 5. The graph of $L(x)$ where $0 \leq x < 0.1$

Since the total length of the perimeter of the triangle $ABC$ is greater than the total length of the perimeter of the right triangle $GHC$, it follows that the total length of the perimeter of the triangle $T$ is greater than two. This is a contradiction. Hence, the regular decagon $R$ is a cover for the family of all triangles of perimeter two.
Next, we note that every cover for the family of all triangles of perimeter two must cover the line segment of length one. Hence, the diagonal of any regular decagon cover for the family of all triangles of perimeter two must have length at least one. The diagonal of the regular decagon $R$ has length one. Therefore, the regular decagon $R$ is a smallest cover for the family of all triangles of perimeter two.

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**References**


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