Common Solution of Split Feasibility, Systems of Equilibrium and Fixed Points Problems in Hilbert Spaces

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Abstract

The purpose of this paper is to introduce an iterative algorithm for finding a common element of the solution sets of split feasibility problems (SFP), systems of equilibrium problems (SEP) and fixed points (for nonexpansive semigroup) problems in Hilbert spaces. We prove that the sequences generated by the proposed algorithm converge weakly to a common element of that solution sets under mild conditions. Our main result improve and extend the recent ones announced by Ceng, Ansari and Yao [1] and many others.

Keywords: Nonexpansive semigroup; Split feasibility problems; Systems of equilibrium problems

1 Introduction

Let $H$ be a real Hilbert space, and let $C$ be a nonempty closed convex subset of $H$. A mapping $T$ of $C$ into itself is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. We denote the set of fixed points of $T$ by $F(T)$. One parameter family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ of mappings of $C$ into itself is called a nonexpansive semigroup on $C$ if it satisfies the following conditions:
\begin{itemize}
    \item[(i)] $T(0)x = x$ for all $x \in C$;
    \item[(ii)] $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
    \item[(iii)] $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
    \item[(iv)] for all $x \in C$, $s \mapsto T(s)x$ is continuous.
\end{itemize}

We denote the set of all common fixed points of $\mathcal{I}$ by $F(\mathcal{I})$, that is, $F(\mathcal{I}) = \{x \in C : T(s)x = x, 0 \leq s < \infty\}$. It is known that $F(\mathcal{I})$ is closed and convex. Later, Baillon and Brezis [2] prove that if $\mathcal{I} = \{T(s) : 0 \leq s < \infty\}$ is a nonexpansive semigroup on $C$, then the continuous scheme
\[ x_t = \frac{1}{t} \int_0^t T(s)x_t ds, \quad t \in (0, 1) \]
converges weakly to a common fixed point of $\mathcal{I}$ (see, for instance Takahashi [3]).

The split feasibility problem (SFP) in Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction was first introduced by Censor and Elfving [4] (see, for instance, [5, 6]). We had known that the SFP can also be used to model the intensity-modulated radiation therapy (see; [7, 8]). In this paper, the SFP is formulated as finding a point $x^*$ with the property
\[ x^* \in C \text{ and } Ax^* \in Q, \] (1.1)

where $C$ and $Q$ are the nonempty closed convex subsets of the infinite-dimensional real Hilbert spaces $H_1$ and $H_2$, respectively, and $A \in B(H_1, H_2)$ (i.e. $A$ is a bounded linear operator from $H_1$ to $H_2$). Recently, there are related works in [5, 7, 9] and the references therein.

In fact, it has been extensively investigated in the literature using the projected Landweber iterative method [10, 11]. Throughout this paper, we assume that the solution set $\Gamma$ of the SFP is nonempty.

Let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \rightarrow \mathbb{R}$ is to find $x \in C$ such that
\[ F(x, y) \geq 0, \forall \ y \in C. \] (1.2)

The set of solutions of (1.2) is denoted by $EP(F)$. Numerous problems in physics, economics and optimizations reduce to find a solution of (1.2) in Hilbert spaces (see, for instance, [12, 13, 14]). Moreover, Combettes, et al. [13] introduced an iterative scheme of finding the best approximation to the solution of equilibrium problem, when $EP(F)$ is nonempty, and proved a strong convergence theorem (see also in [7]-[20]). Let $F_1, F_2 : C \times C \rightarrow \mathbb{R}$ be two monotone bifunction and $\lambda > 0$ is a constant. In 2009, Moudafi [21]
considered the following of a system of equilibrium problem, denote the set of solution of SEP by \( \Omega \), for finding \( (x^*, y^*) \in C \times C \) such that
\[
\begin{align*}
\lambda F_1(x^*, z) + \langle y^* - x^*, x^* - z \rangle & \geq 0, \quad \forall \ z \in C, \\
\lambda F_2(y^*, z) + \langle x^* - y^*, y^* - z \rangle & \geq 0, \quad \forall \ z \in C.
\end{align*}
\]
(1.3)
He also proved the weak convergence theorem of this problem (some related work can be found in \([22, 23]\)).

Motivated and inspired by Ceng, Ansari and Yao \([1]\) and given above paper. The propose of this paper is to introduce and analyze an extragradient method with regularization for finding a common element the solution sets of split feasibility problems (SFP), systems of equilibrium problems (SEP) and fixed points (for nonexpansive semigroup) problems in Hilbert spaces. Combining the regularization method and extragradient method due to Ceng, Ansari and Yao \([1]\), we propose an iterative algorithm for finding an element \( \hat{x} \in F(\mathcal{I}) \cap \Gamma \cap \Omega \). We prove that the sequences generated by the proposed method converge weakly to an element \( \hat{x} \in F(\mathcal{I}) \cap \Gamma \cap \Omega \). However, the technique of the proof of one of the results of this paper is close to that in \([1]\).

2 Preliminaries

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \), and let \( C \) be a closed convex subset of \( H \). We write \( x_n \rightharpoonup x \) to indicate that the sequence \( \{x_n\} \) converges weakly to \( x \) and \( x_n \to x \) to indicate that the sequence \( \{x_n\} \) converges strongly to \( x \). For every point \( x \in H \), there exists a unique nearest point in \( C \), denote by \( P_Cx \), such that
\[
\|x - P_Cx\| = \inf_{y \in C} \|x - y\| \leq \|x - y\| \text{ for all } y \in C.
\]
\( P_C \) is called the metric projection of \( H \) onto \( C \).

Some important properties of projections are gathered in the following proposition.

**Proposition 2.1.** \([1]\) Let \( H \) be a real Hilbert space and let \( C \) be a closed convex subset of \( H \). For given \( x \in H \) and \( z \in C \) then we have the following properties:

(i) \( z = P_Cx \) if and only if \( \langle x - z, y - z \rangle \leq 0 \), for all \( y \in C \);

(ii) \( z = P_Cx \) if and only if \( \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2 \), for all \( y \in C \);

(iii) For all \( y \in H \), \( \langle P_Cx - P_Cy, x - y \rangle \geq \|P_Cx - P_Cy\|^2 \);

**Definition 2.2.** \([24]\) Let \( T \) be a nonlinear operator with domain \( D(T) \subseteq H \) and range \( R(T) \subseteq H \), and let \( \beta > 0 \) and \( v > 0 \) be given constants. The operator \( T \) is called \( v \)-inverse strongly monotone (\( v \)-ism) if
\[
\langle x - y, Tx - Ty \rangle \geq v\|Tx - Ty\|^2, \forall x, y \in D(T).
\]
Definition 2.3. [1] A mapping \( T : H \to H \) is said to be an averaged mapping if it can be written as the average of the identity \( I \) and a nonexpansive mapping, that is,
\[
T \equiv (1 - \alpha)I + \alpha S,
\]
where \( \alpha \in (0, 1) \) and \( S : H \to H \) is nonexpansive. More precisely, when (2.1) holds, we say that \( T \) is \( \frac{1}{2} - \) averaged. Thus firmly nonexpansive mappings (in particular, projections) are \( \frac{1}{2} - \) averaged maps.

Proposition 2.4. [9, 25] Let \( T : H \to H \) be a given mapping.

(i) \( T \) is averaged if and only if the complement \( I - T \) is \( v \)-ism for some \( v > \frac{1}{2} \). Indeed, for \( \alpha \in (0, 1) \), \( T \) is \( \alpha \)-averaged if and only if \( I - T \) is \( \frac{1}{2\alpha} \)-ism.

(ii) The composite of finitely many averaged mappings is averaged. That is, if each of the mappings \( \{T_i\}_{i=1}^N \) is averaged, then so is the composite \( T_1 \circ T_2 \circ \cdots \circ T_N \). In particular, if \( T_1 \) is \( \alpha_1 \)-averaged and \( T_2 \) is \( \alpha_2 \)-averaged, where \( \alpha_1, \alpha_2 \in (0, 1) \), then the composite \( T_1 \circ T_2 \) is \( \alpha \)-averaged, where \( \alpha = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2 \).

The so-called demiclosedness principle for nonexpansive mappings will often be used.

Lemma 2.5. [26] Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and let \( S : C \to C \) be a nonexpansive mapping with \( F(S) \neq \emptyset \). If the sequence \( \{x_n\} \subseteq C \) converges weakly to \( x \) and the sequence \( \{(I - S)x_n\} \) converges strongly to \( y \), then \( (I - S)x = y \); in particular, if \( y = 0 \), then \( x \in F(S) \).

The following elementary result in the real Hilbert spaces is quite well-known.

Lemma 2.6. [26] Let \( H \) be a real Hilbert space, \( x, y \in H \) and \( \lambda \in [0, 1] \). Then
\[
\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.
\]

A set-valued mapping \( T : H \to 2^H \) is called monotone if for all \( x, y \in H \), \( f \in Tx \) and \( g \in Ty \) imply
\[
\langle x - y, f - g \rangle \geq 0.
\]

A monotone mapping \( T : H \to 2^H \) is called maximal if its graph \( G(T) \) is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping \( T \) is maximal if and only if, for \( (x, f) \in H \times H \), \( \langle x - y, f - g \rangle \geq 0 \) for every \( (y, g) \in G(T) \) implies \( f \in Tx \). Let \( F : C \to H \) be a monotone and \( k \)-Lipschitz continuous mapping and let \( N_Cv \) be the normal cone to \( C \) at \( v \in K \), that is,
\[
N_Cv = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}.
\]
Define
\[ T_v = \begin{cases} \quad Fv + N_Cv & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C, \end{cases} \]
Then, \( T \) is maximal monotone and \( 0 \in T_v \) if and only if \( v \in VI(C, F) \); see [27] for more details.

For solving the mixed equilibrium problems for an equilibrium bifunction \( F : C \times C \to \mathbb{R} \), let us assume that \( F \) satisfies the following conditions:
(A1) \( F(x, x) = 0 \) for all \( x \in C \);
(A2) \( F \) is monotone, that is, \( F(x, y) + F(y, x) \leq 0 \) for all \( x, y \in C \);
(A3) for each \( y \in C \), \( x \mapsto F(x, y) \) is weakly upper semicontinuous;
(A4) for each \( x \in C \), \( y \mapsto F(x, y) \) is convex; semicontinuous.

**Lemma 2.7.** [13] Assume that \( F : C \times C \to \mathbb{R} \) satisfies (A1)-(A4). For \( r > 0 \) and \( x \in H \), define a mapping \( T_r : H \to C \) as follows:
\[ T_r(x) = \{ z \in C : F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \forall y \in C \} \]
for all \( z \in H \). Then, the following hold:
(i) \( T_r \) is single-valued;
(ii) \( T_r \) is firmly nonexpansive, i.e., for any \( x, y \in H \), \( \| T_rx - T_ry \|^2 \leq \langle T_rx - T_ry, x - y \rangle \);
(iii) \( F(T_r) = EP(F) \);
(iv) \( EP(F) \) is closed and convex.

**Lemma 2.8.** [28] Let \( H \) be a real Hilbert space and let \( C \) be a closed convex subset of \( H \). Let \( F_1 \) and \( F_2 \) be two mappings from \( C \times C \to \mathbb{R} \) satisfying (A1)-(A4) and let \( T_{1, \lambda} \) and \( T_{2, \mu} \) are defined as in Lemma 2.7 associated to \( F_1 \) and \( F_2 \), respectively. For given \( x^*, y^* \in C \), \((x^*, y^*) \) is a solution of problem (1.3) if and only if \( x^* \) is a fixed point of the mapping \( G : C \to C \) defined by
\[ G(x) = T_{1, \lambda}(T_{2, \mu}x), \quad \forall x \in C \]
where \( y^* = T_{2, \mu}x^* \).

**Lemma 2.9.** (see [29]). Let \( \{a_n\}_{n=1}^\infty \), \( \{b_n\}_{n=1}^\infty \) and \( \{\delta_n\}_{n=1}^\infty \) be sequences of nonnegative real numbers satisfying the inequality
\[ a_{n+1} \leq (1 + \delta_n)a_n + b_n, \forall n \geq 1. \quad (2.2) \]
If \( \sum_{n=1}^\infty \delta_n < \infty \) and \( \sum_{n=1}^\infty b_n < \infty \), then \( \lim_{n \to \infty} a_n \) exists. If, in addition, \( \{a_n\}_{n=1}^\infty \) has a subsequence which converges to zero, then \( \lim_{n \to \infty} a_n = 0 \).
3 Weak convergence theorem

In this section, we prove that the sequences generated by the proposed algorithm converge weakly to an element of nonexpansive semigroup, split feasibility and systems of equilibrium problems in Hilbert spaces. The function \( f : H \to \mathbb{R} \) is a continuous differentiable function with the minimization problem given by

\[
\min_{x \in C} f(x) := \frac{1}{2} \| Ax - P_Q Ax \|^2.
\]  

(3.1)

In [6], Xu considered the following Tikhonov regularized problem

\[
\min_{x \in C} f_\alpha(x) := \frac{1}{2} \| Ax - P_Q Ax \|^2 + \frac{1}{2} \alpha \| x \|^2,
\]  

where \( \alpha > 0 \) is the regularization parameter. The gradient given by

\[
\nabla f_\alpha(x) = \nabla f(x) + \alpha I = A^*(I - P_Q)A + \alpha I
\]

is \((\alpha + \| A \|^2)\)-Lipschitz continuous and \( \alpha \)-strongly monotone (see [1] for the details).

Lemma 3.1. (see [1, 6, 30]). The following holds

(i) \( \Gamma = F(P_C(I - \lambda \nabla f)) = VI(C, \nabla f) \) for any \( \lambda > 0 \), where \( F(P_C(I - \lambda \nabla f)) \) and \( VI(C, \nabla f) \) denote the set of fixed points of \( P_C(I - \lambda \nabla f) \) and the solution set of VIP;

(ii) \( P_C(I - \lambda \nabla f_\alpha) \) is \( \xi \)-averaged for each \( \lambda \in (0, 2/(\alpha + \| A \|^2)) \), where \( \xi = (2 + \lambda(\alpha + \| A \|^2))/4 \).

Theorem 3.2. Let \( C \) be a nonempty closed convex subset in a real Hilbert space \( H \). Let \( F_1 \) and \( F_2 \) be two bifunctions from \( C \times C \to \mathbb{R} \) satisfying (A1) – (A4). Let \( \mu > 0 \) and let \( T_{1, \mu} \) and \( T_{2, \mu} \) be defined as in Lemma 2.7 associated to \( F_1 \) and \( F_2 \), respectively. Let \( \mathcal{S} = \{ S(t) : t > 0 \} \) be a nonexpansive semigroup on \( C \) such that \( F(\mathcal{S}) \cap \Gamma \cap \Omega \neq \emptyset \). Let \( \{ x_n \}, \{ y_n \}, \{ z_n \} \) and \( \{ w_n \} \) be the sequence in \( C \) generated by the following extragradient algorithm:

\[
x_0 = x \in C,
\]

\[
w_n \in C; F_2(w_n, z) + \frac{1}{\mu}(z - w_n, w_n - x_n) \geq 0, \ \forall z \in C,
\]

\[
z_n \in C; F_1(z_n, z) + \frac{1}{\mu}(z - z_n, z_n - w_n) \geq 0, \ \forall z \in C,
\]

\[
y_n = P_C(I - \lambda_n \nabla f_\alpha)z_n,
\]

\[
x_{n+1} = \beta_n x_n + (1 - \beta_n)S(t_n)P_C(x_n - \lambda_n \nabla f_\alpha(y_n)),
\]

where \( \sum_{n=0}^{\infty} \alpha_n < \infty \), \( \{ \lambda_n \} \subset [a, b] \) for some \( a, b \in (0, \frac{1}{\| A \|^2}) \) and \( \{ \beta_n \} \subset [c, d] \) for some \( c, d \in (0, 1) \). Then, the sequences \( \{ x_n \} \) and \( \{ y_n \} \) converge weakly to an element \( \tilde{x} \in F(\mathcal{S}) \cap \Gamma \cap \Omega \).
Proof. 

By Lemma 3.1, we have $P_{C}(I - \lambda \nabla f_{a})$ is $\xi$-averaged for each $\lambda \in (0, \frac{2}{\alpha + \|A\|^2})$, where $\xi = \frac{2 + \lambda(\alpha + \|A\|^2)}{4}$. Hence, by Proposition 2.4, $P_{C}(I - \lambda_{n} \nabla f_{a_{n}})$ is nonexpansive for all $n \geq 0$. Moreover, by the same as in the proof of Theorem 9 in Sombut and Plubtieng [30], we note that

$$\lim_{n \to \infty} \|x_{n} - p\| \text{ exists, for each } p \in F(\mathcal{S}) \cap \Gamma \cap \Omega,$$

and the sequences $\{x_{n}\}, \{y_{n}\}, \{z_{n}\}$ and $\{w_{n}\}$ are bounded. For the last relations, we also obtain that

$$(1 - d)(1 - b^{2}(\alpha_{n} + \|A\|^2)^2)\|x_{n} - y_{n}\|^2 + c(1 - d)\|x_{n} - S(t_{n})l_{n}\|^2$$

$$\leq (1 - \beta_{n})(1 - \lambda_{n}^{2}(\alpha_{n} + \|A\|^2)^2)\|x_{n} - y_{n}\|^2$$

$$+ \beta_{n}(1 - \beta_{n})\|x_{n} - S(t_{n})l_{n}\|^2$$

$$\leq (1 + 2\alpha_{n})\|x_{n} - p\|^2 - \|x_{n+1} - p\|^2$$

$$+ \alpha_{n}\lambda_{n}^{2}\|p\|^2(1 + 2\lambda_{n}^{2}),$$

where $\{\lambda_{n}\} \subset [a, b]$ and $\{\beta_{n}\} \subset [c, d]$. So, from 3.5 and $\alpha_{n} \to 0$, we have

$$\lim_{n \to \infty} \|x_{n} - y_{n}\| = \lim_{n \to \infty} \|x_{n} - S(t_{n})l_{n}\|$$

$$\leq \lim_{n \to \infty} [\|x_{n} - p\| + \|S(t_{n})p - S(t_{n})l_{n}\|] \leq \lim_{n \to \infty} [\|x_{n} - p\| + \|p - l_{n}\|] = 0. \quad (3.6)$$

Hence, we have

$$\lim_{n \to \infty} \|x_{n} - z_{n}\| = 0. \quad (3.7)$$

Put $l_{n} = P_{C}(x_{n} - \lambda_{n} \nabla f_{a_{n}}(y_{n}))$ for each $n \geq 0$. Thus, we have

$$\|y_{n} - l_{n}\| = \|P_{C}(z_{n} - \lambda_{n} \nabla f_{a_{n}}z_{n}) - P_{C}(x_{n} - \lambda_{n} \nabla f_{a_{n}}(y_{n}))\|$$

$$\leq \|(z_{n} - \lambda_{n} \nabla f_{a_{n}}(z_{n})) - (x_{n} - \lambda_{n} \nabla f_{a_{n}}(y_{n}))\|$$

$$= \|z_{n} - x_{n} - (\lambda_{n} \nabla f_{a_{n}}(z_{n}) - \lambda_{n} \nabla f_{a_{n}}(y_{n}))\|$$

$$\leq \|z_{n} - x_{n}\| - (\lambda_{n} \nabla f_{a_{n}}(z_{n}) - \lambda_{n} \nabla f_{a_{n}}(y_{n}))\|$$

$$\leq \|z_{n} - x_{n}\| + \lambda_{n}[\|\nabla f_{a_{n}}(z_{n}) - \nabla f_{a_{n}}(x_{n})\| + \|\nabla f_{a_{n}}(x_{n}) - \nabla f_{a_{n}}(y_{n})\|]$$

$$\leq \|z_{n} - x_{n}\| + \lambda_{n}[\|\alpha_{n} + \|A\|^2\|x_{n} - y_{n}\| + \|\nabla f_{a_{n}}(z_{n}) - \nabla f_{a_{n}}(x_{n})\|]. \quad (3.8)$$

This together with 3.7 and $\alpha_{n} \to 0$, implies that

$$\lim_{n \to \infty} \|y_{n} - l_{n}\| = 0. \quad (3.9)$$
Note that
\[ \|l_n - S(t_n)l_n\| \leq \|l_n - y_n\| + \|y_n - x_n\| + \|x_n - S(t_n)l_n\|. \]
This together with 3.6 and 3.9 implies that
\[ \lim_{n \to \infty} \|l_n - S(t_n)l_n\| = 0. \] (3.10)

Without loss of generality, as in [31], let
\[ \lim_{j \to \infty} \frac{\|l_{n_j} - S(t_{n_j})l_{n_j}\|}{t_{n_j}} = 0. \] (3.11)

Also, from \( \|x_n - l_n\| \leq \|x_n - y_n\| + \|y_n - l_n\| \) we get
\[ \lim_{n \to \infty} \|x_n - l_n\| = 0. \] (3.12)

Since \( \nabla f = A^*(I - P_C)A \) is Lipschitz conditions, we have
\[ \lim_{n \to \infty} \|\nabla f(y_n) - \nabla f(l_n)\| = 0. \]

As \( \{x_n\} \) is a bounded sequence, there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) that converges weakly to some \( \hat{x} \).

Next, we show that \( \hat{x} \in F(3) \), that is, \( \hat{x} = S(t)\hat{x} \) for a fixed \( t > 0 \). It easy to see that
\[
\|l_{n_j} - S(t)\hat{x}\| \leq \sum_{m=0}^{[t/t_{n_j}]-1} \|S((m+1)(t_{n_j})l_{n_j}) - S((m)t_{n_j})l_{n_j}\| + \|S[t/t_{n_j}]l_{n_j} - S[t/t_{n_j}]\hat{x}\|
\]
\[
\leq \left[ \frac{t}{t_{n_j}} \right] \|l_{n_j} - S(t_{n_j})l_{n_j}\| + \|l_{n_j} - \hat{x}\| + \|S(t - [t/t_{n_j}]t_{n_j})\hat{x} - \hat{x}\|
\]
\[
\leq \left[ \frac{t}{t_{n_j}} \right] \|l_{n_j} - S(t_{n_j})l_{n_j}\| + \|t_{n_j} - \hat{x}\| + \sup\{\|S(q)\hat{x} - \hat{x}\|; 0 \leq q \leq t_{n_j}\}. \]

\[ (3.13) \]

This fact, together with 3.11 and property (iv) for the semigroup, implies that
\[ \lim_{j \to \infty} \sup \|l_{n_j} - S(t)\hat{x}\| \leq \lim_{j \to \infty} \sup \|l_{n_j} - \hat{x}\|. \]

As every Hilbert space satisfies Opial’s condition, we have \( S(t)\hat{x} = \hat{x} \). Therefore \( \hat{x} \in F(3) \).

Next, we show that \( \hat{x} \in \Gamma \). Since \( \|x_n - l_n\| \to 0 \) and \( \|y_n - l_n\| \to 0 \), it is know that \( l_n \to \hat{x} \) and \( y_n \to \hat{x} \). Let
\[
Tv = \begin{cases} \nabla f(v) + N_Cv & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C, \end{cases} \]

\[ (3.14) \]
where $N_C v = \{ w \in H_1 : \langle v - u, w \rangle \geq 0, \forall u \in C \}$. Then, $T$ is maximal monotone and $0 \in T v$ if and only if $v \in VI(C, \nabla f)$; see [32], for more details. Let $(v, w) \in G(T)$. Then, we have $w \in T v = \nabla f(v) + N_C v$ and hence $w - \nabla f(v) \in N_C v$. So, we have $\langle v - u, w - \nabla f(v) \rangle \geq 0, \forall u \in C$. On the other hand, from $l_n = P_C(x_n - \lambda_n \nabla f_\alpha(y_n))$ and $v \in C$, we have

$$\langle x_n - \lambda_n \nabla f_\alpha(y_n) - l_n, l_n - v \rangle \geq 0,$$

and hence

$$\langle v - l_n, \frac{l_n - x_n}{\lambda_n} + \nabla f_\alpha(y_n) \rangle \geq 0.$$

Therefore, from $w - \nabla f(v) \in N_C v$ and $l_n \in C$, we have

$$\langle v - l_n, w \rangle \geq \langle v - l_n, \nabla f(v) \rangle \geq \langle v - l_n, \frac{l_n - x_n}{\lambda_n} + \nabla f_\alpha(y_n) \rangle = \langle v - l_n, \nabla f(v) \rangle - \langle v - l_n, \frac{l_n - x_n}{\lambda_n} + \nabla f(y_n) \rangle - \alpha_n \langle v - l_n, y_n \rangle = \langle v - l_n, \nabla f(l_n) - \nabla f(y_n) \rangle - \langle v - l_n, \frac{l_n - x_n}{\lambda_n} \rangle - \alpha_n \langle v - l_n, y_n \rangle \geq \langle v - l_n, \nabla f(l_n) - \nabla f(y_n) \rangle - \langle v - l_n, \frac{l_n - x_n}{\lambda_n} \rangle - \alpha_n \langle v - l_n, y_n \rangle. \tag{3.15}$$

Hence, we obtain that $\langle v - \hat{x}, w \rangle \geq 0$ as $i \to \infty$. Since $T$ is maximal monotone, we have $\hat{x} \in T^{-1}0$, and hence $\hat{x} \in VI(C, \nabla f)$. Thus, it is clear that $\hat{x} \in \Gamma$.

Next, we show that $\hat{x} \in \Omega$. Let $G$ be a mapping which is defined as in Lemma 2.8, thus we have

$$\|z_n - G(z_n)\| = \|T_{1,\mu} T_{2,\mu} x_n - G(z_n)\| = \|G(x_n) - G(z_n)\| \leq \|x_n - z_n\| \tag{3.16}$$

and hence

$$\|x_n - G(x_n)\| \leq \|x_n - z_n\| + \|z_n - G(z_n)\| + \|G(z_n) - G(x_n)\| \leq 3\|x_n - z_n\|. \tag{3.17}$$

As $n \to \infty$, we get $\|x_n - G(x_n)\| \to 0$. From $\lim_{n \to \infty} \|x_n - z_n\| = 0$ and $z_{n_j} \to \hat{x}$, we get $x_{n_j} \to \hat{x}$. According to Lemma 2.5 and Lemma 2.8, we have $\hat{x} \in \Omega$. Therefore, we have $\hat{x} \in F(\mathcal{S}) \cap \Gamma \cap \Omega$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \to \hat{x}$. We may show that $\hat{x} = \bar{x}$, suppose not. Since $\lim_{n \to \infty} \|x_n - \hat{x}\|$ exists for all $\hat{x} \in F(\mathcal{S}) \cap \Gamma \cap \Omega$, it
follows by the Opial’s condition that
\[
\lim_{n \to \infty} \|x_n - \hat{x}\| = \liminf_{i \to \infty} \|x_{i_n} - \hat{x}\| < \liminf_{i \to \infty} \|x_{i_n} - \hat{x}\| = \lim_{n \to \infty} \|x_n - \hat{x}\| = \liminf_{j \to \infty} \|x_{j_n} - \hat{x}\| < \liminf_{j \to \infty} \|x_{j_n} - \hat{x}\| = \lim_{n \to \infty} \|x_n - \hat{x}\|. \tag{3.18}
\]
It is a contradiction. Thus, we have \(\hat{x} = \tilde{x}\). This implies that 
\(x_n \rightharpoonup \tilde{x} \in F(\mathcal{S}) \cap \Gamma \cap \Omega\). Further, from 
\(\|x_n - y_n\| \to 0\), it follows that 
\(y_n \rightharpoonup \tilde{x}\) and so \(z_n, w_n\). This complete the prove. \(\square\)

Theorem 3.2 extends the extragradient method due to Nadezhkina and Takahashi [33].

**Theorem 3.3.** Let \(C\) be a nonempty closed convex subset in a real Hilbert space \(H\). Let \(F_1\) and \(F_2\) be two bifunctions from \(C \times C \to \mathbb{R}\) satisfying (A1) – (A4). Let \(\mu > 0\) and let \(T_{1,\mu}\) and \(T_{2,\mu}\) be defined as in Lemma 2.7 associated to \(F_1\) and \(F_2\), respectively. Let
\(\mathcal{S} = \{S(t) : t > 0\}\) be a nonexpansive semigroup on \(C\) such that \(F(\mathcal{S}) \cap \Gamma \cap \Omega \neq \emptyset\). Let
\(\{x_n\}, \{y_n\}, \{z_n\}\) and \(\{w_n\}\) be the sequence in \(C\) generated by the following extragradient algorithm :

\[
\begin{cases}
    x_0 = x \in C, \\
    w_n \in C; F_2(w_n, z) + \varphi(z) - \varphi(w_n) + \frac{1}{\mu}(z - w_n, w_n - x_n) \geq 0, \quad \forall \, z \in C, \\
    z_n \in C; F_1(z_n, z) + \varphi(z) - \varphi(z_n) + \frac{1}{\mu}(z - z_n, z_n - w_n) \geq 0, \quad \forall \, z \in C, \\
    y_n = P_C(I - \lambda_n \nabla f)z_n, \\
    x_{n+1} = \beta_n x_n + (1 - \beta_n)T(t_n)P_C(x_n - \lambda_n \nabla f(y_n)),
\end{cases}
\]

where \(\{\lambda_n\} \subset [a, b]\) for some \(a, b \in (0, \frac{1}{\|A\|^2})\) and \(\{\beta_n\} \subset [c, d]\) for some \(c, d \in (0, 1)\). Then, the sequences \(\{x_n\}\) and \(\{y_n\}\) converge weakly to an element \(\hat{x} \in F(\mathcal{S}) \cap \Gamma \cap \Omega\).

Utilizing Theorem 3.2, we have the following results in the setting of real Hilbert spaces.
Setting \(F_2 = 0\) in Theorem 3.2, we have the following result.

**Corollary 3.4.** Let \(C\) be a nonempty closed convex subset in a real Hilbert space \(H\). Let \(F_1\) and \(F_2\) be two bifunctions from \(C \times C \to \mathbb{R}\) satisfying (A1) – (A4). Let \(\mu > 0\) and let \(T_{1,\mu}\) and \(T_{2,\mu}\) be defined as in Lemma 2.7 associated to \(F_1\) and \(F_2\), respectively. Let
Let $\mathcal{S} = \{S(t) : t > 0\}$ be a nonexpansive semigroup on $C$ such that $F(\mathcal{S}) \cap \Gamma \cap \Omega \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequence in $C$ generated by the following extragradient algorithm:

$$
\begin{align*}
x_0 &= x \in C, \\
z_n &\in C; F_1(z_n, z) + \frac{1}{n} \langle z - z_n, z_n - x_n \rangle \geq 0, \quad \forall \ z \in C, \\
y_n &= P_C(I - \lambda_n \nabla f_{\alpha_n})z_n, \\
x_{n+1} &= \beta_n x_n + (1 - \beta_n) T(t_n) P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)),
\end{align*}
$$

where $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\|A\|_2})$ and $\{\beta_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $\hat{x} \in F(\mathcal{S}) \cap \Gamma \cap \Omega$.

Setting $F_1 = F_2 = 0$ in Theorem 3.2, we have the following result.

**Corollary 3.5.** Let $C$ be a nonempty closed convex subset in a real Hilbert space $H$. Let $\mathcal{S} = \{S(t) : t > 0\}$ be a nonexpansive semigroup on $C$ such that $F(\mathcal{S}) \cap \Gamma \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequence in $C$ generated by the following extragradient algorithm:

$$
\begin{align*}
x_0 &= x \in C, \\
y_n &= P_C(I - \lambda_n \nabla f_{\alpha_n})x_n, \\
x_{n+1} &= \beta_n x_n + (1 - \beta_n) T(t_n) P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)),
\end{align*}
$$

where $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\|A\|_2})$ and $\{\beta_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $\hat{x} \in F(\mathcal{S}) \cap \Gamma$.

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**References**


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