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A New Modified Simplex Method to Solve Quadratic Fractional Programming Problem and Compared it to a Traditional Simplex Method by Using Pseudoaffinity of Quadratic Fractional Functions

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Abstract

In this paper, we defined a new modified simplex method to solve quadratic fractional programming problem (QFPP) and suggested an algorithm for it. The algorithm of usual simplex method is also reported. The special case for this problem was solved by Converting objective function to pseudoaffinity of quadratic fractional functions (PQFF) to a linear programming problem to be solved by simplex method. Then the result is compared with a result, which obtained by new modified simplex method. These methods demonstrated by numerical examples. This work confirms that our techniques is valid and can be used to solve this particular type of QFPP.

Keywords Modified Simplex Method, PQFF, QFPP, Simplex Method

1 Introduction

The quadratic fractional programming problems (QFPP) are the topic of great importance in nonlinear programming. They are useful in many fields such as production planning, financial and corporative planning, health care and hospital planning. In various applications of nonlinear programming, one often encounters the problem in which the ratio of given two functions is to be maximized or minimized. Several methods to solve such problems are proposed in (1962)[4], their method depends on transforming this Linear Fractional Programming Problem (LFPP) to an equivalent Linear Program. Archana Khurana and Arora (2011) studied for solving a QFP when some of its constraints are homogeneous [1]. Moreover, Quadratic fractional programming problems with quadratic constraints have been addressed by Fukushima and Hayashi (2008) [7]. Abdulrahim, B., K. (2011) studies on Solving Quadratic Programming Problem with Extreme Point [2]. Also M. Biggs at (2005) worked on Nonlinear Optimization with Financial Applications [8]. [9] Nejmaddin, Maher they have study on a solving quadratic fractional programming problem by using the Wolfe's method and a modified simplex approach (2013). In order to extend this work, we consider a special case problem in which the objective function is QF (quadratic fractional) but contains linear constraints. The problem will solve by a new modified simplex method. In addition, the special case problem will be solved by simplex method after convert the objective function to the pseudoaffinity function. Both results will be compared to exam validity.

2 Definitions and theorems related with this work:

Definition 2.1: Linear programming (LP).

LP is a technique for the optimization of a linear objective function, subject to linear equality and linear inequality constraints. Given a polyhedron and a real-valued affine function defined on this polyhedron, a linear programming method will find a point on the polyhedron where this function has the smallest (or largest) value if such point exists, by searching through the polyhedron vertices. [13]

A linear program is an optimization problem of the form

Maximize $c^t x$

Subject to

$$Ax = b$$

And

$$x \geq 0$$

Where $c \in R^n$, $b \in R^n$ and A $\in R^m X^n$. [5]

Definition 2.2: Quadratic programming

If the optimization problem assumes the form

$$\text{max. z (or min. z)} = \alpha + C^T x + x^T G x$$

subject to:

$$Ax \begin{pmatrix} \leq \\ \geq \\ = \end{pmatrix} b$$

$$x \geq 0$$

Where $A = (a_{ij})_{m \times n}$, matrix of coefficients, $\forall i = 1,2, \ldots, m$ and $= 1,2, \ldots, n$, $b = (b_1, b_2, \ldots, b_m)^T$, $x = (x_1, x_2, \ldots, x_n)^T$, $C^T = (c_1, c_2, \ldots, c_n)^T$, and $G = (g_{ij})_{n \times n}$ is a positive definite or positive semi-definite symmetric square matrix, moreover the constraints are linear and the objective function is quadratic. Such optimization problem is said to be a quadratic programming (QP) problem. [3]

Theorem 2.3: Fundamental theorem of linear programming.

If the optimal value of the objective function in a linear programming problem exists, then that value (known as the optimal solution) must occur at one or more of the extreme points of the feasible region; see [6]

3 Mathematical form of QFPP

The mathematical form of this type of problems is given as follows:

$$max. z = \frac{(c^T x + \delta + \frac{1}{2} x^T G x)}{(C^T x + \gamma)}$$

Subject to:

$$Ax \begin{bmatrix} \geq \\ \leq \\ = \end{bmatrix} b$$
$$x \geq o$$

Where G are $(n \times n)$ matrix of coefficients with G are symmetric matrixes. All vectors are assumed to be column vectors unless transposed(T), where x is an n-dimensional vector of decision variables, c, C is the n-dimensional vector of constants, b is n-dimensional vector of constants. γ, δ are scalars.

In this work the problem that has objective function is tried to be solved has the following form:

$$max. z = \frac{(c_1{}^T x+\alpha)(c_2{}^T x+\beta)}{(C^T x+\gamma)}$$

Subject to:

$$Ax \begin{bmatrix} \geq \\ \leq \\ = \end{bmatrix} b$$
$$x \geq o$$

A is m × n matrix ,all vectors are assumed to be column vectors unless transposed (T). where x is an n-dimensional vector of decision variables, c_1, c_2, C are the n-dimensional vector of constants, b is n-dimensional vector of constants, α, β,γ are scalars.

4 Pseudoaffinity of quadratic fractional functions.

In this section we are going to characterize the pseudoaffinity of quadratic fractional functions of the following kind:

$$f(x) = \frac{(c^T x+\delta+\frac{1}{2}x^T Gx)}{(C^T x+\gamma)} \qquad (1)$$

defined on the set $X = \{x \in \Re^n : C^T x + \gamma > 0\}$, where $G \neq 0$ is a $n \times n$ symmetric matrix, $c, x, C \in \Re^n$, $b \neq 0$, and $\delta, \gamma \in \Re$. Note that being G symmetric, it is $G \neq 0$ if and only if $v_0(G) \leq n - 1$. [11]

Corollary 4.1: Consider function f defined in (1) and suppose that there exist $b_0, q_0, p_0 \in \Re$, $\alpha \neq 0$, such that f can be written in the following form:

$$f(x) = b_0 C^T x + q_0 + \frac{b_0 p_0}{(C^T x + p_0)}$$

i) If $p_0 \leq 0$ then f is pseudoaffine on X.

ii) If $p_0 > 0$ then f is pseudoaffine on X_1={$x \in \Re^n : C^T x + p_0 > \sqrt{p_0}$} and X_2={$x \in \Re^n : 0 < C^T x + p_0 < \sqrt{p_0}$}. [11]

In our studies we take special case where $p_0 = 0$, then the function $f = b_0 C^T x + q_0$ is pseudoaffine on X, and linear functions by adding constraints then it can solve it by simplex method, which is shown in numerical examples section.

5 Simplex method

It has been possible to obtain the graphical solution to the LP problem of more than two variables. The analytic solution is also not possible because the tools of analysis are not well suited to handle inequalities. In such case, a simple and most widely used simplex method is adopted which was developed by Dantzig in 1947. The simplex method provides an algorithm (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the fundamental theorem of linear programming. [12].

Many different methods have been proposed to solve linear programming problems, but simplex method has proved to be the most effective. This method is applicable to any problem that can be formulated in terms of linear objective function, subject to a set of linear constraints.[10]

5.1 Solution for linear programming problem by simplex method

Back to a reference [12], page (60) to see process of solution of usual simplex method.

5.2 The Simplex Algorithm

The simplex technique has proved to be the most effective in solving linear programming problems. In the simplex method, an initial basic solution (extreme point) should be found firstly.
Then, we proceed to an adjacent extreme point. We continue this process until an optimal Solution is reached. See usual Simplex Algorithm in reference [10].

6 Modified Simplex Method Development

Simplex method is developed by Dantzig in (1947). The Simplex method provides a systematic algorithm which consists of moving from one basic feasible solution (on vertex) to another in prescribed manner such value of the objective function is improved. This procedure of jumping from vertex to vertex is repeated. If the objective function is improved at each jump, then no basis can ever be repeated and there is no need go to back to the vertex because it already being covered. Since the number of vertices is finite, the process must lead to the optimal vertex in a finite number of steps. The Simplex algorithm is an iterative (step by step) procedure for solving linear programming problems, and it consists of:

i) Having a trail basic feasible solution to constraint equations.
ii) Testing whether is an optimal solution.
iii) Improving the first trial solution by a set of rules, and repeating the process till an optimal solution is obtained, for more details see [12].

6.1 A new modified simplex method for solving quadratic fractional programming problem

This section deals with the solution of the quadratic fractional programming problem by the method exactly similar to Simplex Technique in linear programming. This method can be successfully adapted to high speed computational. We can apply this method if the constraints of the problem are linear function. i.e. our problem is of the form:

$$Max. z\ (or\ Min. z) = \frac{(c^T x+\delta+\frac{1}{2}x^T G x)}{(C^T x+\gamma)} = \frac{(c_1{}^T x+\alpha)(c_2{}^T x+\beta)}{(C^T x+\gamma)}$$

Subject to:

$$Ax\begin{pmatrix}\leq\\ \geq\\ =\end{pmatrix}b$$

$$x \geq 0$$

i) A is m × n matrix;

ii) x, c, c_1, c_2, C are n × 1 column vectors;

iii) b is m × 1 column vectors;

iv) $\alpha, \beta, \gamma, \delta$ are scalars and prime (T) denoted the transpose of the vector.

v) Where G is $(n \times n)$ matrix of coefficients with G are symmetric matrixes. All vectors are assumed to be column vectors unless transposed(T).

Here it is assumed that $(c_1{}^T x + \alpha)(c_2{}^T x + \beta)$ and $(C^T x + \gamma)$ are positive all feasible solutions, the set of all feasible bounded, and closed convex polyhedron. Also, at least two distinct feasible solutions exist.

Using Modified Simplex Method developed to solve the numerical example to apply simplex process, first we find Δk_1^i , Δk_2 from the coefficient of numerator and denominator of objective function respectively, by using the following formula:

$\Delta k_1^{ij} = c_{Bi}\, x_j - c_{ij}$, $i = 1,2,\ j = 1,2, \dots, m + n$
$\Delta k_2^{1j} = C_B x_j - C_{1j}$, $j = 1,2, \dots, m + n$
$z_1^1 = c_{B1} x_B + \alpha, z_1^2 = c_{B2} x_B + \beta, z_2 = C_B x_B + \gamma$

$Z_1 = z_1^1 z_1^2,\ Z_2 = z_2,$
$\Delta\xi_{1j} = z_1^2 \Delta k_1^{1j} + z_1^1 \Delta k_1^{2j}$
$\Delta\xi_{2j} = \Delta k_2^{1j}$
$Z = Z_1 / Z_2$

In this approach we define the formula to find Δ_j from $Z_1, Z_2, \Delta\xi_{1j}$ and $\Delta\xi_{2j}$ as follows:

$$\Delta_j = Z_2 \Delta\xi_{1j} - Z_1 \Delta\xi_{2j}$$

Here c_{1j}, c_{2j}, C_{1j} are the coefficients of the basic and non-basic variables in the objective function and c_{B1}, c_{B2}, C_B are the coefficients of the basic variables in the objective function, $j = 1,2, \dots, m + n$.And will show the process of the solution in the section of numerical examples.

6.2 Algorithm for the new modified simplex method

An algorithm to solve QFPP by Modified Simplex Method which can be summarized as follows:

Step1: Write the standard form of the problem, by introducing slack and artificial variables to constraints, and write starting Simplex table.

Step2: Calculate the Δ_j by the following formula $\Delta_j = Z_2 \Delta\xi_{1j} - Z_1 \Delta\xi_{2j}$, then write it in the starting Simplex table.

Step3: Find the solution by using Simplex process.

Step4: Check the solution for feasibility in step3, if it is feasible then go to step5, otherwise use dual Simplex method to remove infeasibility.

Step5: Check the solution for optimality if all $\Delta_j \geq 0$, then the solution is optimal, otherwise go to step3.

7 Numerical Example and Results:

In this section we tested several numerical examples but only two of them are presented. Consider the following Quadratic Fractional Programming Problem as:

Example 1: Maximize $Z = \frac{8x_1^2+24x_1x_2+18x_2^2-2}{6x_1+9x_2+3}$

Subject to:

$$x_1 + 3x_2 \leq 5$$
$$2x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Solution: (New modified simplex method)

Solving the problem by a new modified simplex method is considered, detail process of the solution is given as follows.

Max. $Z = \frac{8x_1^2+24x_1x_2+18x_2^2-2}{6x_1+9x_2+3} = \frac{(4x_1+6x_2-2)(2x_1+3x_2+1)}{(6x_1+9x_2+3)}$

Subject to:

$$x_1 + 3x_2 \leq 5$$
$$2x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

The first table of the new modified simplex method is given as follows:

B.v	C_B	c_{B_1}	c_{B_2}	X_B	x_1	x_2	x_3	x_4	Min ratio
				c_{1j}	4	6	0	0	
				c_{2j}	2	3	0	0	
				C_{1j}	6	9	0	0	
x_2	9	6	3	$\frac{5}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$	0	5
x_4	0	0	0	$\frac{1}{3}$	$\frac{5}{3}$	0	$\frac{-1}{3}$	1	0.2
	$z_1^1 = 8$			Δk_1^{1j}	-2	0	2	0	
	$z_1^2 = 6$			Δk_1^{2j}	-1	0	1	0	
	$z_2 = 18$			Δk_2^{1j}	-3	0	3	0	
	$Z_1 = 48$			$\Delta\xi_{1j}$	-20	0	20	0	
	$Z_2 = 18$			$\Delta\xi_{2j}$	-3	0	3	0	
	$Z = \frac{Z_1}{Z_2} = \frac{8}{3}$			Δ_j	-216	0	216	0	

B.v	C_B	c_{B_1}	c_{B_2}	X_B	x_1	x_2	x_3	x_4	Min ratio
				c_{1j}	4	6	0	0	
				c_{2j}	2	3	0	0	
				C_{1j}	6	9	0	0	
x_3	0	0	0	5	1	3	1	0	1.6
x_4	0	0	0	2	2	1	0	1	2
	$z_1^1 = -2$			Δk_1^{1j}	-4	-6	0	0	
	$z_1^2 = 1$			Δk_1^{2j}	-2	-3	0	0	
	$z_2 = 3$			Δk_2^{1j}	-6	-9	0	0	
	$Z_1 = -2$			$\Delta\xi_{1j}$	0	0	0	0	
	$Z_2 = 3$			$\Delta\xi_{2j}$	-6	-9	0	0	
	$Z = \frac{Z_1}{Z_2} = \frac{-2}{3}$			Δ_j	-12	-18	0	0	

In the above table the feasible solution is $x_1 = 0,\ x_2 = 0, x_3 = 5,\ x_4 = 2$ and $Max. Z = -2/3$ this solution is not optimal, because all Δ_j not greater than or equal 0, then we find next feasible solution, as follows:

Since $\Delta_j = -18$, then we select x_2 to be entering vector and we find minimum ratio where min ratio = min $\{\frac{V_B}{x_2}; x_i > 0$ and min {1.6, 2} =1.6 in the first row will be x_3 the outing vector. Pivot element is at row 1 column 2. By applying Simplex technique we get:

For testing optimality solution must all $\Delta_j \geq 0$ but here all Δ_j not greater than zero, and then the solution is not optimal. Repeat the same approach to find next feasible solution, and get:

				c_{1j}	4	6	0	0	
				c_{2j}	2	3	0	0	
				C_{1j}	6	9	0	0	
B.v	C_B	c_{B_1}	c_{B_2}	X_B	x_1	x_2	x_3	x_4	Min ratio
x_2	9	6	3	1.6	0	1	0.4	-0.2	
x_1	6	4	2	0.2	1	0	-0.2	0.6	
	$z_1^1 = 8.4$			Δk_1^{1j}	0	0	1.6	1.2	
	$z_1^2 = 6.2$			Δk_1^{2j}	0	0	0.8	0.6	
	$z_2 = 18.6$			Δk_2^{1j}	0	0	2.4	1.8	
	$Z_1 = 52.08$			$\Delta\xi_{1j}$	0	0	16.64	12.48	
	$Z_2 = 18.6$			$\Delta\xi_{2j}$	0	0	2.4	1.8	
	$Z = \frac{Z_1}{Z_2} = 2.8$			Δ_j	0	0	184.512	138.384	

Then the solution is $x_1 = 0.2,\ x_2 = 1.6$ and $Max. Z = 2.8$, since all $\Delta_j \geq 0$ then the solution is optimal.

Solution: (Simplex method by using pseudoaffine function)

Solving the same problem by simplex method after convert the objective function to pseudoaffine function as follows.

Max. $Z = \frac{8x_1^2+24x_1x_2+18x_2^2-2}{6x_1+9x_2+3}$ where $G = \begin{bmatrix} 8 & 12 \\ 12 & 18 \end{bmatrix}$ $, c^t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C^t = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ $, \delta = -2, \gamma = 3$. Then we get. $\text{Max. } Z = \frac{4}{3}x_1 + 2x_2 - \frac{2}{3}$ is pseudoaffine function by Corollary 4.1 (i) where $p_0 = 0$. After finding the values of each individual objective functions by simplex method with used the same constraint, the following result is found. $x_1 = 0.2$, $x_2 = 1.6$ and $\text{Max. } Z = 2.8$.

Example 2: Maximize $Z = \frac{4x_1^2+12x_1x_2+8x_2^2+4x_1+4x_2}{4x_1+8x_2+4}$

Subject to:

$$-2x_1 + x_2 \leq 3$$
$$4x_1 + 2x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

Solution: (new modified simplex method).

Maximize $Z = \frac{(2x_1+2x_2)(2x_1+4x_2+2)}{4x_1+8x_2+4}$

Subject to:

$$-2x_1 + x_2 \leq 3$$
$$4x_1 + 2x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

Now solving the problem by using new modified simplex method after 4 steps we construct initial table as follows:

B.v	C_B	c_{B_1}	c_{B_2}	X_B	x_1	x_2	x_3	x_4	Min ratio
				c_{1j}	2	2	0	0	
				c_{2j}	2	4	0	0	
				C_{1j}	4	8	0	0	
x_3	0	0	0	3	-2	1	1	0	—
x_4	0	0	0	8	4	2	0	1	2
	$z_1^1 = 0$			Δk_1^{1j}	-2	-2	0	0	
	$z_1^2 = 2$			Δk_1^{2j}	-2	-4	0	0	
	$z_2 = 4$			Δk_2^{1j}	-4	-8	0	0	
	$Z_1 = 0$			$\Delta\xi_{1j}$	-4	-4	0	0	
	$Z_2 = 4$			$\Delta\xi_{2j}$	-4	-8	0	0	
	$Z = \frac{Z_1}{Z_2} = 0$			Δ_j	-16	-16	0	0	

After two iteration, we obtained the result in the following table:

B.v	C_B	c_{B_1}	c_{B_2}	X_B	x_1	x_2	x_3	x_4	Min ratio
				c_{1j}	2	2	0	0	
				c_{2j}	2	4	0	0	
				C_{1j}	4	8	0	0	
x_2	8	2	4	$\frac{7}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{4}$	—
x_1	4	2	2	$\frac{1}{4}$	1	0	$\frac{-1}{4}$	$\frac{1}{8}$	2
	$z_1^1 = 7.5$			Δk_1^{1j}	0	0	$\frac{1}{2}$	$\frac{3}{4}$	
	$z_1^2 = 16.5$			Δk_1^{2j}	0	0	$\frac{3}{2}$	$\frac{5}{4}$	
	$z_2 = 33$			Δk_2^{1j}	0	0	3	$\frac{5}{2}$	
	$Z_1 = 123.75$			$\Delta\xi_{1j}$	0	0	19.5	21.75	
	$Z_2 = 33$			$\Delta\xi_{2j}$	0	0	3	2.5	
	$Z = \frac{Z_1}{Z_2} = 3.75$			Δ_j	0	0	272.25	408.375	

Then the solution is $x_1 = \frac{1}{4},\ x_2 = \frac{7}{2}$ and $Max. Z = 3.75$, since all $\Delta_j \geq 0$ then the solution is optimal.

Solution: (Simplex method by using pseudoaffine function)

Solveing the same problem by simplex method after convert the objective function to pseudoaffine function as follows.

Max. Z $= \frac{4x_1^2+12x_1x_2+8x_2^2+4x_1+4x_2}{4x_1+8x_2+4}$ where $G = \begin{bmatrix} 4 & 6 \\ 6 & 8 \end{bmatrix}$ $, c^t = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, C^t = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ $, \delta = 0, \gamma = 4$ .Then we get. Max. Z $= x_1 + x_2$ is pseudoaffine function by Corollary 4.1 (i) where $p_0 = 0$.After finding the values of each individual objective functions by simplex method with the use of same constraint, the following result is found $x_1 = \frac{1}{4},\ x_2 = \frac{7}{2}$ and $Max. Z = 3.75$.

8 Comparison of the numerical results is shown in the following table

Examples	New Modified Simplex Method	Simplex method by using pseudoaffine function
Ex.1	$x_1 = 0.2, \quad x_2 = 1.6,$ $Max. Z = 2.8$	$x_1 = 0.2,\ x_2 = 1.6$, $Max. Z = 2.8$
Ex.2	$x_1 = \frac{1}{4},\ x_2 = \frac{7}{2}$, $Max. Z = 3.75$	$x_1 = \frac{1}{4},\ x_2 = \frac{7}{2}$, $Max. Z = 3.75$

In the above table, we compare the result. It is notice that value of objective function in ex.(1) , ex(2) they have same results when it solved by new modified simplex method and when it solved by the simplex method after convert the objective function pseudoaffine function.

9 Conclusion

This paper used new modified simplex method and simplex method after convert objective function to pseudoaffine function to found the maximum value of QFPP, The comparisons of these methods are based on the value of the objective function, the study found that max.z resulted of those two method are same. Therefor we can solve special case of QFPP by this two method under our Technique and algorithm. Consequently reliable results have been found.

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