On a Class of Nonlinear Partial Differential Equations

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Abstract
The purpose of this paper is to obtain radial upper bounds for the solutions of nonlinear partial differential inequalities of the form

\[ \Delta u \geq P(r)f(u) \]

in the annulus \( \rho B_R(0) \), centered at the origin and with radii \( \rho, R(\rho < R) \). We give criteria for the existence, nonexistence and uniqueness of solutions of certain \( 2m^{th} \) order nonlinear Dirichlet type problems.

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1 Introduction

Many results have appeared in the literature on the subject for solutions of the nonlinear partial differential equation

\[ \Delta u = f(u), \] (1)

or, more generally, the differential inequality

\[ \Delta u \geq f(u). \] (2)

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where $\Delta$ is the $N$-dimensional Laplace operator. In [3, 4, 7] conditions on $f$ were obtained in order that a radially symmetric bound for solutions of (2) may exist. The most general conditions on $f$ for which radially symmetric bounds for the solutions of (2) may exist are

$$f(u) > 0, f'(u) \geq 0 \text{ for } -\infty < u < \infty \quad (3)$$

$$\int_0^\infty \left[ \int_0^u f(t) dt \right]^{-\frac{1}{2}} du < \infty. \quad (4)$$

In fact, if $f(u) > 0$ and $f'(u) \geq 0$, condition (4) is both necessary and sufficient. It was shown by Osserman [7] that if $\varphi(r)$ is a spherically symmetric solution of (1) i.e. a solution of the ordinary differential equation

$$\varphi''(r) + \frac{N-1}{r} \varphi'(r) = f(\varphi), \quad (5)$$

for which $\varphi'(0) = 0$ and $\varphi(r) \to \infty$ as $r \to R$ then if $u$ is any solution of (1) for $r \leq R$, we have $u(x_1, x_2, \ldots, x_N) \leq \varphi(r)$ at each point.

In [6] Nehari used the result of Osserman to obtain explicit upper and lower bounds for solutions of (1) and (2) for certain class of functions $f$.

In [2] the authors showed that the function $\nu$ defined by

$$\nu = \frac{P(r)(R^2 - r^2)}{8R^2} = \int_0^\infty \frac{dt}{f(t)}, \quad (6)$$

is a radial upperbound for the solutions $u(x_1, x_2)$ of

$$\Delta u \geq P(r)f(u),$$

in an open disc $B_R(0)$ of radius $R$ with center at the origin, where $f$ is positive, monotone increasing, $C^1$ function satisfying

$$f'(u) \int_u^\infty \frac{dt}{f(t)} \leq 1, \quad (7)$$

and $P(r)$ is assumed to be positive.

Radial upper bound was also obtained for the solutions of $\Delta \omega \geq \alpha f(\omega)$, $\alpha$ is a positive constant in an open ball $B_R$ where $f$ is positive, monotone increasing, $C^1$ function and, in addition, satisfies the condition
On a class of nonlinear partial differential equations

\[ f'(\omega) \int_{\omega}^{\infty} \frac{dt}{f(t)} \leq \frac{N + 2}{4}, \quad N \geq 3. \quad (8) \]

Here by a slight extension of Osserman’s lemma, and the theorems A and B below, we obtain radial upper bounds for the solutions of some inequalities in the annulus \( \rho B_R(0) \) centered at the origin and with radii \( \rho, R(\rho < R) \). In section 3, we give conditions for the existence, nonexistence and uniqueness of solutions of \( 2m^{th} \) order nonlinear Dirichlet type boundary value problems.

We shall use the following results proved in [2]:

**Theorem A:** Let \( \omega \) be a positive \( C^2 \) function for which \( \Delta \omega \leq Q(x)\omega^k \), \( k > 1 \), for some positive \( Q(x) \). If \( g \) is a positive \( C^1 \) function for which

\[ g'(s) \int_{s}^{\infty} \frac{dt}{g(t)} \leq \frac{k}{k - 1}, \quad (9) \]

and \( \nu \) defined by

\[ \int_{\nu}^{\infty} \frac{dt}{g(t)} = \frac{1}{(k - 1)\omega^{k-1}}, \quad (10) \]

then

\[ \Delta \nu \leq Q(x)g(\nu). \quad (11) \]

**Theorem B:** Let \( \omega \) be a positive \( C^2 \) function for which \( \Delta \omega \leq Q(x)e^\omega \) for some positive function \( Q(x) \). If \( g(x) \) is a positive \( C^1 \) function such that

\[ g'(s) \int_{s}^{\infty} \frac{dt}{g(t)} \leq 1, \quad (12) \]

and \( \nu \) defined by

\[ \int_{\nu}^{\infty} \frac{dt}{g(t)} = e^{-\omega} \quad (13) \]

then

\[ \Delta \nu \leq Q(x)g(\nu). \quad (14) \]
2 Bound for Solutions

Let \( \rho B_R \) denote the open ring between two concentric spheres of radius \( \rho \) and \( R \) (\( \rho < R \)) respectively with center at the origin in Euclidean \( N \)-space and let \( r \) denote the distance from the origin to an arbitrary point \( x = (x_1, x_2, x_3, \ldots, x_N) \) in \( \rho B_R \).

We first prove the following Lemma:

**Lemma 1.** Let \( y(x_1, x_2, x_3, \ldots, x_N) \) be a \( C^2 \) function in \( \rho B_R \) defined by

\[
y = \left( \frac{\sqrt{(N + 2\lambda - 2)(N + 6\lambda - 2)}R^{2\lambda}}{\sqrt{2}(r^{2\lambda} - \rho^{2\lambda})(R^{2\lambda} - r^{2\lambda})} \right)^{\frac{N + 2\lambda - 2}{4\lambda}}, \tag{15}\]

where \( \lambda \geq 1 \) is a constant, then

\[
\Delta y \leq r^{4\lambda - 2} y^{\frac{N + 10\lambda - 2}{N + 2\lambda - 2}}. \tag{16}\]

**Proof.** Consider the function \( \nu \) defined by

\[
\nu = \frac{1}{(r^{2\lambda} - \rho^{2\lambda})^\alpha(R^{2\lambda} - r^{2\lambda})^\alpha}, \tag{17}\]

where \( \alpha \) is a constant to be determined later. We let \( x \) denote one of the variables \( x_k \) and differentiate (17) twice with respect to \( x \). This results in

\[
\nu_x = -\frac{2\alpha \lambda x^2 r^{2\lambda - 2}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha} + \frac{2\alpha \lambda x r^{2\lambda - 2}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha}
\]

\[
\nu_{xx} = -\frac{2\alpha x^2 r^{2\lambda - 2} + 4\alpha x^2 \lambda (\lambda - 1) r^{2\lambda - 4}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha} + \frac{4\alpha x^2 \lambda^2 (\alpha + 1) r^{4\lambda - 4}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha}
\]

\[
-\frac{2\alpha \lambda x r^{2\lambda - 2} + 4\alpha x^2 \lambda (\lambda - 1) r^{2\lambda - 4}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha} + \frac{4\alpha x^2 \lambda^2 (\alpha + 1) r^{4\lambda - 4}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha}
\]

\[
+ \frac{4\alpha x^2 \lambda (\lambda - 1) r^{2\lambda - 2}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha} + \frac{4\alpha x^2 \lambda^2 (\alpha + 1) r^{4\lambda - 4}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha}
\]

Summing over all \( x \), we get

\[
\Delta \nu = -\frac{2N\alpha \lambda x r^{2\lambda - 2} + 4\alpha \lambda (\lambda - 1) r^{2\lambda - 2}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha} + \frac{4\alpha x^2 \lambda^2 (\alpha + 1) r^{4\lambda - 4}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha}
\]

\[
-\frac{2N\alpha \lambda x r^{2\lambda - 2} + 4\alpha \lambda (\lambda - 1) r^{2\lambda - 2}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha} + \frac{4\alpha x^2 \lambda^2 (\alpha + 1) r^{4\lambda - 4}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha}
\]

\[
+ \frac{4\alpha x^2 \lambda (\lambda - 1) r^{2\lambda - 2}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha} + \frac{4\alpha x^2 \lambda^2 (\alpha + 1) r^{4\lambda - 4}}{(r^{2\lambda} - \rho^{2\lambda})^{\alpha + 1}(R^{2\lambda} - r^{2\lambda})^\alpha}
\]
Combining 1st, 3rd, 4th and 6th terms on the right, we obtain

\[ \Delta \nu = \frac{2N\lambda \alpha r^{2\lambda} - 2 - 4\lambda(\lambda - 1)}{(r^2 - \rho^2)^{\alpha + 1}(\lambda - 2\lambda)\alpha + 1} + \frac{\alpha(\alpha + 1)\lambda^2 r^{4\lambda - 2}\lambda + 2 + 8\lambda^2 \lambda^2 r^{4\lambda - 2}}{(r^2 - \rho^2)^{\alpha + 2}(\lambda - 2\lambda)\alpha + 2}. \]

Using the fact that \( \rho < r < R \), we get,

\[ \Delta \nu \leq \frac{2\lambda \alpha [N + 2(\lambda - 1) - 4\lambda\alpha] r^{4\lambda - 2}}{(r^2 - \rho^2)^{\alpha + 1}(\lambda - 2\lambda)\alpha + 1} + \frac{8\lambda(\alpha + 1)\lambda^2 r^{4\lambda - 2} R^{4\lambda}}{(r^2 - \rho^2)^{\alpha + 2}(\lambda - 2\lambda)\alpha + 2}. \]  

(18)

Now choose \( \alpha = \frac{N + 2\lambda - 2}{4\lambda} \), then by (17), (18) reduces to

\[ \Delta \nu \leq \frac{(N + 2\lambda - 2)(N + 6\lambda - 2)}{2} r^{4\lambda - 2} R^{4\lambda} y^{1 + 8\lambda} \frac{N + 2\lambda - 2}{N + 2\lambda - 2}. \]  

(19)

Next, by the change of variable

\[ \nu = \left[ \frac{2}{(N + 2\lambda - 2)(N + 6\lambda - 2)} \right]^{\frac{N + 2\lambda - 2}{8\lambda}} \frac{y \sqrt{N + 2\lambda - 2}}{R^{\frac{N + 2\lambda - 2}{2}}}, \]  

(20)

the inequality (19) becomes

\[ \Delta y \leq r^{4\lambda - 2} \frac{N + 2\lambda - 2}{y^{N + 2\lambda - 2}}, \]

where

\[ y = \left( \frac{\sqrt{(N + 2\lambda - 2)(N + 6\lambda - 2) R^{2\lambda}}}{\sqrt{2}(r^2 - \rho^2)^{\lambda - 2\lambda}} \right)^{\frac{N + 2\lambda - 2}{4\lambda}}, \]

which proves the assertion. \( \square \)

As a consequence of Lemma 1, we have the following result:

**Theorem 1.** If \( u = u(x_1, x_2, x_3, \ldots, x_N) \) satisfies the inequality

\[ \Delta u \geq r^{4\lambda - 2} f(u), \quad \lambda \geq 1 \]  

(21)

where \( f(u) \) is positive, monotone increasing, \( C^1 \) function on \((0, \infty)\) such that
\[ f'(u) \int_u^\infty \frac{dt}{f(t)} \leq 1 + \frac{N + 2\lambda - 2}{8\lambda}, \]  
(22)

then the radial function \( \nu \) defined by

\[ \int_\nu^\infty \frac{dt}{f(t)} = \frac{(r^{2\lambda} - \rho^{2\lambda})(R^{2\lambda} - r^{2\lambda})}{4\lambda R^{4\lambda}(N + 6\lambda - 2)}, \]  
(23)

is such that \( u \leq \nu \) at each point in \( \rho < r < R \).

**Proof.** With \( k = 1 + \frac{8\lambda}{N + 2\lambda - 2} \), \( Q(x) = r^{4\lambda} - 2 \) and \( g \) replaced by \( f \) in theorem A, we conclude, by lemma 1 that

\[ \Delta \nu \leq r^{4\lambda - 2} f(\nu), \]

for \( \nu \) defined by

\[ \int_\nu^\infty \frac{dt}{f(t)} = \frac{(R^{2\lambda} - r^{2\lambda})(r^{2\lambda} - \rho^{2\lambda})}{4\lambda (N + 6\lambda - 2) R^{4\lambda}}. \]

Since \( \nu'(0) = 0 \) when \( \lambda > 1 \) and \( \nu \to \infty \) as \( r \to R \) or \( r \to \rho \), it follows by an extension of Osserman’s Lemma that \( u \leq \nu \) at each point of \( \rho B_R \). \( \square \)

**Lemma 2.** If \( w = w(x_1, x_2) \) is a \( C^2 \) function in the punctured disc \( \rho B_R \) defined by

\[ w = 2\ell n \left( \frac{4\sqrt{2k} R^{2k}}{(r^{2k} - \rho^{2k})(R^{2k} - r^{2k})} \right), \]  
(24)

where \( k \geq 1 \) is a constant, then

\[ \Delta w \leq r^{4k - 2} e^w, \]  
(25)

at every point of \( \rho B_R \).

**Proof.** Consider the function \( \nu \) defined by

\[ e^\nu = \frac{A}{(r^{2k} - \rho^{2k})\alpha (R^{2k} - r^{2k})\alpha}, \]  
(26)

where \( A \) and \( \alpha \) are constants to be determined later. We let \( x \) denote one of the variables \( x_1, x_2 \) and differentiate (26) twice with respect to \( x \). This results in
\[ e^\nu \nu_x = A \left( -\frac{2k \alpha r^{2k-2}}{(r^{2k} - \rho^{2k})^{\alpha+1}(R^{2k} - r^{2k})^\alpha} + \frac{2k \alpha r^{2k-2}}{(r^{2k} - \rho^{2k})^\alpha(R^{2k} - r^{2k})^{\alpha+1}} \right), \]  

(27)

\[ e^\nu \nu_{xx} + e^\nu \nu_x^2 = A \left( -\frac{2k \alpha r^{2k-2} + 4k \alpha (k-1)x^2 r^{2k-4}}{(r^{2k} - \rho^{2k})^{\alpha+1}(R^{2k} - r^{2k})^\alpha} + \frac{4\alpha(\alpha+1)k^2 x^2 r^{4k-4}}{(r^{2k} - \rho^{2k})^\alpha(R^{2k} - r^{2k})^{\alpha+1}} \right. 

\[ \left. - \frac{4\alpha^2 k^2 x^2 r^{4k-4}}{(r^{2k} - \rho^{2k})^{\alpha+1}(R^{2k} - r^{2k})^\alpha} + \frac{2k \alpha r^{2k-2} + 4k \alpha (k-1)x^2 r^{2k-4}}{(r^{2k} - \rho^{2k})^\alpha(R^{2k} - r^{2k})^{\alpha+1}} \right) - \frac{4\alpha^2 k^2 x^2 r^{4k-4}}{(r^{2k} - \rho^{2k})^{\alpha+1}(R^{2k} - r^{2k})^\alpha}. \]  

(28)

Using (26) and (27), we write (28) as

\[ \nu_{xx} = -\frac{2k \alpha r^{2k-2} + 4k \alpha (k-1)x^2 r^{2k-4}}{(r^{2k} - \rho^{2k})} + \frac{2k \alpha r^{2k-2} + 4k \alpha (k-1)x^2 r^{2k-4}}{(R^{2k} - r^{2k})} + \frac{4\alpha^2 k^2 x^2 r^{4k-4}}{(r^{2k} - \rho^{2k})^2} + \frac{4\alpha^2 k^2 x^2 r^{4k-4}}{(R^{2k} - r^{2k})^2}. \]

Summing over all \( x \), we get, for \( N = 2 \)

\[ \Delta \nu = \frac{4k^2 \alpha r^{2k-2} \rho^{2k}}{(r^{2k} - \rho^{2k})^2} + \frac{4k^2 \alpha r^{2k-2} R^{2k}}{(R^{2k} - r^{2k})^2}. \]  

(29)

Using the fact that \( \rho < r < R \), we have

\[ \Delta \nu \leq \frac{16k^2 \alpha r^{4k-2} R^{4k}}{(r^{2k} - \rho^{2k})^2(R^{2k} - r^{2k})^2}. \]  

(30)

By (26) it reduces to

\[ \Delta \nu \leq \frac{16k^2 \alpha r^{4k-2} R^{4k} e^{2\nu}}{A^{2/\alpha}}. \]  

(31)

Making the change of variable \( w = \frac{2\nu}{\alpha} \), we obtain

\[ \Delta w \leq \frac{32k^2 \alpha r^{4k-2} R^{4k} e^w}{A^{2/\alpha}}. \]  

(32)
Now choose \( A = (32k^2R^{4k})^{\frac{a}{2}} \), then
\[
\Delta w \leq r^{4k-2} e^w,
\]
where
\[
w = 2\ln \left( \frac{4\sqrt{2}kR^{2k}}{(r^{2k} - \rho^{2k})(R^{2k} - r^{2k})} \right),
\]
which proves the Lemma 2.

Next, as a consequence of Lemma 2, Osserman’s Lemma and theorem B, in a manner similar to theorem 1, one can easily deduce the following result:

**Theorem 2.** If \( u(x_1, x_2) \) satisfies
\[
\Delta u \geq r^{4k-2} f(u), \quad k > 1
\]
where \( f \) is positive, monotone increasing, \( C^1 \) function satisfying
\[
f'(u) \int_u^\infty \frac{dt}{f(t)} \leq 1,
\]
then the function \( \nu \) defined by
\[
\int _\nu \frac{dt}{f(t)} = \frac{(r^{2k} - \rho^{2k})^2(R^{2k} - r^{2k})^2}{32k^2R^{4k}},
\]
is a radial bound for \( u \) in the punctured disc \( \rho B_R \).

We, now extend Lemma 2 and theorem 2 to \( N \)-dimensions. We prove

**Lemma 3.** We \( w = w(x_1, x_2, x_3, \ldots, x_N) \) be a \( C^2 \) function in \( \rho B_R \) such that
\[
w = 2\ln \left( \frac{R^{2k}\sqrt{4k(N + 6k - 2)}}{(r^{2k} - \rho^{2k})(R^{2k} - r^{2k})} \right), \quad N \geq 2
\]
where \( k \geq 1 \) is a constant, then
\[
\Delta w \leq r^{4k-2} e^w,
\]
On a class of nonlinear partial differential equations

Proof. We define the function \( \nu = \nu(x_1, x_2, x_3, \ldots, x_N) \) as in (26) and differentiate twice with respect to \( x \), then we get from (28) with the help of (26)

\[
\nu_{xx} + \nu_x^2 = - \frac{2k\alpha r^{2k-2} + 4\alpha(k-1)r^{2k-4}x^2}{(r^{2k} - \rho^{2k})} + \frac{4\alpha(\alpha+1)k^2 x^2 r^{4k-4}}{(r^{2k} - \rho^{2k})^2} + \frac{4\alpha^2 x^2 r^{4k-4}}{(r^{2k} - \rho^{2k})^2}.
\]

Hence,

\[
\nu_{xx} + \frac{\nu_x^2}{4} \leq - \frac{2k\alpha r^{2k-2} + 4\alpha(k-1)r^{2k-4}x^2}{(r^{2k} - \rho^{2k})} + \frac{4\alpha(\alpha+1)k^2 x^2 r^{4k-4}}{(r^{2k} - \rho^{2k})^2} + \frac{4\alpha^2 x^2 r^{4k-4}}{(r^{2k} - \rho^{2k})^2}.
\]

Summing over all \( x \) and using (27), we get

\[
\Delta \nu \leq \frac{2kN\alpha r^{2k-2}(-R^{2k} + 2r^{2k} - \rho^{2k}) + 4\alpha(k-1)r^{2k-2}(-R^{2k} + 2r^{2k} - \rho^{2k}) - 6k^2 \alpha^2 r^{4k-2}}{(r^{2k} - \rho^{2k})^2(R^{2k} - r^{2k})} + \frac{3\alpha^2 + 4\alpha)k^2 r^{4k-2}(R^{2k} - 2R^{2k} - 2r^{2k} - \rho^{2k})}{(r^{2k} - \rho^{2k})^2(R^{2k} - r^{2k})^2}.
\]

In view of the fact \( \rho < r < R \), it reduces to

\[
\Delta \nu \leq \frac{2k\alpha(N + 2(k - 1) - 3k\alpha)r^{4k-2}}{(r^{2k} - \rho^{2k})(R^{2k} - r^{2k})} + \frac{\alpha(3\alpha + 4)2k^2 R^{4k} r^{4k-2}}{(r^{2k} - \rho^{2k})^2(R^{2k} - r^{2k})^2}. \tag{40}
\]

Now choose \( \alpha = \frac{N + 2k - 2}{3k} \), then

\[
\Delta \nu \leq \frac{2(N + 2k - 2)(N + 6k - 2)R^{4k} r^{4k-2}}{3(r^{2k} - \rho^{2k})^2(R^{2k} - r^{2k})^2}. \tag{41}
\]

By (26) it reduces to

\[
\Delta \nu \leq \frac{2}{3}(N + 2k - 2)(N + 6k - 2)R^{4k} r^{4k-2} \frac{e^{2\nu}}{A^{2/\alpha}}. \tag{42}
\]

By change of variable \( \omega = \frac{2\nu}{\alpha} \), we obtain

\[
\Delta \omega \leq 4k(N + 6k - 2)R^{4k} r^{4k-2} \frac{e^{\omega}}{A^{2/\alpha}}. \tag{43}
\]
Next, let $A = [R^{4k}4k(N + 6k - 2)]^{\frac{N+2k-2}{6k}}$, then

$$\Delta \omega \leq r^{4k-2}e^\omega,$$

where

$$\omega = 2\ln \left( \frac{\sqrt{4k(N + 6k - 2)}R^{2k}}{(r^{2k} - \rho^{2k})(R^{2k} - r^{2k})} \right).$$

This completes the proof of Lemma 3.

In a manner similar to the proof of theorem 2, by (37), theorem B and Osserman’s Lemma, one could derive the following result:

**Theorem 3.** If the function $u = u(x_1, x_2, x_3, \ldots, x_N)$ satisfies

$$\Delta u \geq r^{4k-2}f(u), \quad k \geq 1$$

where $f$ is positive, monotone increasing, $C^1$ function satisfying

$$f'(u) \int_u^\infty \frac{dt}{f(t)} \leq 1,$$

then the function $\nu$ defined by

$$\int_\nu^\infty \frac{dt}{f(t)} = \frac{(r^{2k} - \rho^{2k})^2(R^{2k} - r^{2k})^2}{4kR^{4k}(N + 6k - 2)}, \quad (N \geq 2)$$

is such that $u \leq \nu$ at each point of $\rho B_R$.

**Remark:** For $N = 2$, (46) does not include (35).

### 3 Dirichlet type Problems

Consider the nonlinear Dirichlet type problem

$$\Delta^n u = (-1)^m P(r)e^{au} \quad \text{in } B_R(0),$$

$$u = 0 \quad \text{on } \partial B_R(0),$$

where $B_R(0) \subset \mathbb{R}^N$ is a open ball of radius $R$ centered at 0, $a > 0$ is a constant, the function $P(r) > 0$ is such that

...
\[ \Delta^m (\ell n P(r)) = 0 \quad (49) \]

and \( \Delta^m \) is the \( m \)th iterate of \( N \)-dimensional Laplace operator \( \Delta \).

First, we write the Equation (47) as

\[ \Delta^m u = (-1)^m e^{au + \ell n P(r)}, \]

and let

\[ au + \ell n P(r) = \nu, \quad (50) \]

then, in view of (49), the problem (47), (48) reduces to

\[ \Delta^m \nu = (-1)^m a e^{\nu} \quad \text{in} \ B_R(0), \quad (51) \]

and

\[ \nu = \ell n P(r) \quad \text{on} \ \partial B_R(0). \quad (52) \]

Now by theorem 4 [8] the solution of (51) is given by

\[ \nu = m \ell n \left[ \left( \frac{(2m)!}{a} \right)^{\frac{1}{m}} \frac{4R^2}{(R^2 + r^2)^2} \right]. \]

Hence, by (50)

\[ u = \frac{m}{a} \ell n \left[ \left( \frac{(2m)!}{aP(r)} \right)^{\frac{1}{m}} \frac{4R^2}{(R^2 + r^2)^2} \right]. \]

It can be checked or verified that the solution \( u \) of (47) can be written

\[ u = \frac{m}{a} \ell n \frac{b}{[P(r)]^{\frac{1}{m}}} \left( \frac{1 + \left( \frac{a}{2m} \right)^{\frac{1}{m}} \frac{br^2}{4} \right)^2, \quad (53) \]

where \( b \) is a constant. Thus

\[ e^{au} = \frac{b}{(P(r))^{\frac{1}{m}}} \left( 1 + \left( \frac{a}{2m} \right)^{\frac{1}{m}} \frac{br^2}{4} \right)^2. \quad (54) \]
Since \( u = 0 \) when \( r = R \), we conclude that the constant \( b \) is a root of
\[
\left( \frac{a}{(2m)!} \right)^\frac{2}{m} \frac{R^4}{16} b^2 + \left( \frac{a}{(2m)!} \right)^\frac{1}{m} \frac{R^2}{2} - \frac{1}{(P(R))^{\frac{1}{m}}} \right) b + 1 = 0,
\]
amely,
\[
b = \frac{8((2m)!)^\frac{2}{m}}{a^{2/m}R^4} \left\{ \left( \frac{1}{(P(R))^{\frac{1}{m}}} - \left( \frac{a}{(2m)!} \right)^\frac{1}{m} \frac{R^2}{2} \right) \pm \sqrt{\frac{1}{(P(R))^{\frac{1}{m}}} - \left( \frac{a}{(2m)!P(R)} \right)^{\frac{1}{m}} R^2} \right\}.
\]
Hence, the problem \((47), (48)\) has

(1) No solution if \( R^2(P(R))^{\frac{1}{m}} > (\frac{(2m)!}{a})^{\frac{1}{m}} \)

(2) One solution if \( R^2(P(R))^{\frac{1}{m}} = (\frac{(2m)!}{a})^{\frac{1}{m}} \)

(3) Two solutions if \( R^2(P(R))^{\frac{1}{m}} < (\frac{(2m)!}{a})^{\frac{1}{m}} \).

In particular, if \( m = 1 \), then the problem
\[
\Delta u + P(r)e^{au} = 0 \quad \text{in } B_R(0), \quad u = 0 \quad \text{on } \partial B_R(0),
\]
has

(1) No solution if \( R^2P(R) > \frac{2}{a} \)

(2) One solution if \( R^2P(R) = \frac{2}{a} \)

(3) Two solutions if \( R^2P(R) < \frac{2}{a} \)

\[\square\]

**Remark:** If \( P(R) = K \) (Const.) and \( a = 1 \), then the corresponding result in [8] is a special case of this result.

In Case \( m = 2 \), the Dirichlet type problem for biharmonic functions
\[
\Delta^2 u = P(r)e^{au} \quad \text{in } B_R(0)
\]
\[
u = 0 \quad \text{on } \partial B_R(0)
\]
has
(1) No solution if \( R^2(P(R))^{\frac{1}{2}} > \left(\frac{4}{a}\right)^{\frac{1}{2}} \)

(2) one solution if \( R^2(P(R))^{\frac{1}{2}} = \left(\frac{4}{a}\right)^{\frac{1}{2}} \)

(3) Two solutions if \( R^2(P(R))^{\frac{1}{2}} < \left(\frac{4}{a}\right)^{\frac{1}{2}} \).

Next, we consider the problem

\[
\Delta^k u = (-1)^k bu^{\frac{N+2k}{2}} & \text{ in } B_R(0) \\
u = \phi(R) & \text{ on } \partial B_R(0)
\]

where \( b > 0 \) is a constant and \( N > 2k \).

It is well-known [1] that the solution of (59) is given by

\[
u = \left(\frac{R^k \sqrt{(N-2k)(N-2(k-1)) \cdots (N-2)(N+2)\cdots (N+2(k-1))}}{b(R^2 + r^2)^{k/2}}\right)^{\frac{N-2k}{4}},
\]

which can be written as

\[
u = \left(\frac{(N-2k) \cdots (N+2(k-1))}{b}\right)^{\frac{1}{k}} \frac{1}{R^2(1 + \frac{r^2}{R^2})^2} \cdot \frac{N-2k}{4}.
\]

Further, for a constant \( d \), it can be checked that

\[
u = \left(\frac{d}{1 + \frac{r^2 db^{\frac{1}{k}}}{\{(N-2k) \cdots (N+2(k-1))\}^{\frac{1}{k}}}}\right)^{\frac{N-2k}{4}}.
\]

Using the boundary condition \( u = \phi(R) \) when \( r = R \), we find that \( d \) is a root of

\[
\left(1 + \frac{R^2 db^{\frac{1}{k}}}{\{(N-2k) \cdots (N+2(k-1))\}^{\frac{1}{k}}}\right)^2 = \frac{d}{\phi(R)}^{\frac{N-2k}{4}},
\]

or,

\[
R^4 b^{\frac{2k}{N-2k}} d^2 \left\{\frac{2R^2 b^{\frac{1}{k}}}{\{(N-2k) \cdots (N+2(k-1))\}^{\frac{1}{k}}} - \frac{1}{\phi R^{\frac{N-2k}{4}}(R)}\right\} d + 1 = 0.
\]
Thus,

\[
d = \frac{1}{2R^4} \left\{ \left( \frac{(N - 2k)\cdots(N + 2(k - 1))}{b} \right)^{\frac{2}{k}} \right\} \left[ \frac{1}{\phi(R)^{\frac{2}{N-2k}}} - \frac{2R^2b^{\frac{k}{2}}}{\{(N - 2k)\cdots(N + 2(k - 1))\}^{\frac{k}{2}}} \right] \pm \sqrt{\left( \frac{1}{\phi(R)^{\frac{2}{N-2k}}} - \frac{4R^2b^{\frac{k}{2}}}{\{(N - 2k)\cdots(N + 2(k - 1))\}} \right)^{\frac{k}{4}} \frac{4}{b} \phi^{\frac{4}{N-2k}}(R)} \right) \right]^{\frac{1}{k}}
\]

Hence, the problem (59), (60) has

(i) No solution if \( R^2\phi^{\frac{4}{N-2k}}(R) > \left\{ \left(\frac{(N - 2k)\cdots(N + 2(k - 1))}{b}\right)^{\frac{1}{k}} \right\} \)

(ii) one solution if \( R^2\phi^{\frac{4}{N-2k}}(R) = \left\{ \left(\frac{(N - 2k)\cdots(N + 2(k - 1))}{b}\right)^{\frac{1}{k}} \right\} \)

(iii) Two solutions if \( R^2\phi^{\frac{4}{N-2k}}(R) < \left\{ \left(\frac{(N - 2k)\cdots(N + 2(k - 1))}{b}\right)^{\frac{1}{k}} \right\} \)

In particular, if \( k = 1 \), then the problem

\[
\Delta u + bu^{\frac{N+2}{2}} = 0 \quad \text{in } B_R(0), \quad N \geq 3
\]

\[
u(R) = \phi(R) \quad \text{on } \partial B_R(0),
\]

has

(i) No solution if \( R^2\phi^{\frac{4}{N-2}}(R) > \frac{N(N-2)}{4b} \),

(ii) one solution if \( R^2\phi^{\frac{4}{N-2}}(R) = \frac{N(N-2)}{4b} \),

(iii) Two solutions if \( R^2\phi^{\frac{4}{N-2}}(R) < \frac{N(N-2)}{4b} \).

In case \( k = 2 \), then the problem

\[
\Delta^2 u = bu^{\frac{N+4}{4}} \quad \text{in } B_R(0), \quad N \geq 5
\]

\[
u = \phi(R) \quad \text{on } \partial B_R(0)
\]

has
On a class of nonlinear partial differential equations

(i) No solution if \( R^2 \phi^{N-4} (R) > \frac{\sqrt{(N-4)(N-2)N(N+2)}}{4\sqrt{b}} \),

(ii) one solution if \( R^2 \phi^{N-4} (R) = \frac{\sqrt{(N-4)(N-2)N(N+2)}}{4\sqrt{b}} \),

(iii) Two solutions if \( R^2 \phi^{N-4} (R) < \frac{\sqrt{(N-4)(N-2)N(N+2)}}{4b} \).

References


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